Moment equations and multi-species BGK and LBO operators

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1 General Moment Equations

Each species in a multi-component plasma is described by the Boltzmann equation which describes the temporal evolution of the particle distribution function in a six dimensional spatial and velocity space and evolves under the influence of collisions and electromagnetic forces. With the distribution function for species s, $f^s(\mathbf{x},\mathbf{v},t)$, defined such that $f^s(\mathbf{x},\mathbf{v},t)d\mathbf{x}d\mathbf{v}$ is the number of particles in a phase-space volume element $d\mathbf{x}d\mathbf{v}$, the Boltzmann equation may be written as

$$\frac{\partial f^s}{\partial t} + v_j \frac{\partial f^s}{\partial x_j} + \frac{q_s}{m_s}(E_j + \epsilon_{kmj} v_k B_m)\frac{\partial f^s}{\partial v_j} = \mathcal{H}^s \tag{1}$$

Here $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic flux density, q_s and m_s are the charge and mass of the plasma species and ϵ_{kmj} is the completely anti-symmetric pseudo-tensor which is defined to be ± 1 for even/odd permutations of $(1,2,3)$ and zero otherwise. Summation over repeated indices is assumed.

General moment equations can be derived from the Boltzmann equation. To do this we adopt the following definition for the moments

$$P^{(n)}_{i_1 i_2 \ldots i_n} \equiv m \int_{-\infty}^{\infty} c_{i_1} c_{i_2} \ldots c_{i_n} f d^3\mathbf{v} \tag{2}$$

and generalized friction from collisions

$$R^{(n)}_{i_1 i_2 \ldots i_n} \equiv m \int_{-\infty}^{\infty} c_{i_1} c_{i_2} \ldots c_{i_n} \mathcal{H} d^3\mathbf{v} \tag{3}$$

Here, the species index is dropped and $\mathbf{c} \equiv \mathbf{v} - \mathbf{u}$. With this definition, for example, $P^{(0)} = mn(\mathbf{x},t)$, where $n(\mathbf{x},t)$ is the number density, $P^{(1)}_i = 0$, $P^{(2)}_{ij} = P_{ij}$, where P_{ij} is the pressure tensor and $P^{(3)}_{ijk} = Q_{ijk}$, where Q_{ijk} is the heat flux tensor, etc.

The non-conservative form of the general moment equations are as follows. For $n=1$ we have

$$\partial_t u_{i_1} + u_j \partial_j u_{i_1} + \frac{1}{P^{(0)}} \partial_j P^{(2)}_{j i_1} = \frac{q}{m}(E_{i_1} + B_m u_k \epsilon_{k m i_1}) + \frac{1}{P^{(0)}} R^{(1)}_{i_1}. \tag{4}$$

For $n \neq 1$ (including $n=0$) we have

$$\begin{aligned} \partial_t P^{(n)}_{i_1 i_2 \ldots i_n} - \frac{1}{P^{(0)}} \partial_j P^{(2)}_{j[i_1} P^{(n-1)}_{i_2 \ldots i_n]} + \partial_j P^{(n+1)}_{j i_1 i_2 \ldots i_n} + \partial_j u_j P^{(n)}_{i_1 i_2 \ldots i_n} + \partial_j u_{[i_1} P^{(n)}_{i_2 \ldots i_n]j} + u_j \partial_j P^{(n)}_{i_1 i_2 \ldots i_n} \\ = \frac{q}{m} B_m \epsilon_{km[i_1} P^{(n)}_{i_2 \ldots i_n k]} + R^{(n)}_{i_1 i_2 \ldots i_n} - \frac{1}{P^{(0)}} R^{(1)}_{[i_1} P^{(n-1)}_{i_2 \ldots i_n]} \end{aligned} \tag{5}$$

In these equations square brackets around indices represent the minimal sum over permutations of free indices needed to yield completely symmetric tensors[1].

[1] For example, $u_{[i}E_{j]} = u_i E_j + u_j E_i$. In general, as the moments are themselves symmetric, one usually only needs to cyclically permute the free indices and sum to get a symmetric expression.

The general conservative form of the moment equations can be derived. For this the following total velocity moments are defined

$$\mathcal{P}^{(n)}_{i_1 i_2 \ldots i_n} \equiv m \int_{-\infty}^{\infty} v_{i_1} v_{i_2} \ldots v_{i_n} f d^3\mathbf{v} \tag{6}$$

and generalized total velocity friction from collisions

$$\mathcal{R}^{(n)}_{i_1 i_2 \ldots i_n} \equiv m \int_{-\infty}^{\infty} v_{i_1} v_{i_2} \ldots v_{i_n} \mathcal{H} d^3\mathbf{v} \tag{7}$$

With these the general conservative form of the equations are

$$\partial_t \mathcal{P}^{(n)}_{i_1 \ldots i_n} + \partial_j \mathcal{P}^{(n+1)}_{j i_1 \ldots i_n} = \frac{q}{m}(E_{[i_1} \mathcal{P}^{(n-1)}_{i_2 \ldots i_n]} + B_m \epsilon_{mj[i_1} \mathcal{P}^{(n)}_{i_2 \ldots i_n j]}) + \mathcal{R}^{(n)}_{i_1 \ldots i_n}. \tag{8}$$

2 Ten- and five-moment equations

Consider the special case of the ten-moment equations in which one only retains the evolution of the moments upto $\mathcal{P}^{(3)}_{ijk} \equiv \mathcal{Q}_{ijk}$. These equations are

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x_j}(n u_j) = 0 \tag{9}$$

$$m \frac{\partial}{\partial t}(n u_i) + \frac{\partial \mathcal{P}_{ij}}{\partial x_j} = nq\left(E_i + \epsilon_{ijk} u_j B_k\right) \tag{10}$$

$$\frac{\partial \mathcal{P}_{ij}}{\partial t} + \frac{\partial \mathcal{Q}_{ijk}}{\partial x_k} = nq u_{[i} E_{j]} + \frac{q}{m} \epsilon_{[ikl} \mathcal{P}_{kj]} B_l \tag{11}$$

where $\mathcal{P}^{(2)}_{ij} \equiv \mathcal{P}_{ij}$ is the pressure-tensor in the lab-frame. Note that the pressure tensor and heat-flux tensor in lab-frame can be written as

$$\mathcal{P}_{ij} = P_{ij} + nm u_i u_j \tag{12}$$

$$\mathcal{Q}_{ijk} = Q_{ijk} + u_{[i} \mathcal{P}_{jk]} - 2nm u_i u_j u_k. \tag{13}$$

Now, the total energy is defined as $\mathcal{E} \equiv \mathcal{P}_{ii}/2$, hence we have

$$\mathcal{E} \equiv \frac{1}{2}\mathcal{P}_{ii} = \frac{3}{2}p + \frac{1}{2} mn\mathbf{u}^2 \tag{14}$$

where $p = P_{ii}/3$ it the fluid scalar pressure. Hence, taking (half) the trace of Eq. (11) gives the evolution equation of the total energy

$$\frac{\partial \mathcal{E}}{\partial t} + \frac{1}{2}\frac{\partial \mathcal{Q}_{iik}}{\partial x_k} = nq\mathbf{u} \cdot \mathbf{E} \tag{15}$$

where from Eq. (13) we have

$$\frac{1}{2}\mathcal{Q}_{iik} = q_k + u_k(p + \mathcal{E}) + u_i \Pi_{ik} \tag{16}$$

where $q_k \equiv Q_{iik}/2$ is the *heat-flux* vector and $\Pi_{ij} = P_{ij} - p\delta_{ij}$ is the viscous stress tensor.

3 Multi-species BGK operator

For a plasma with multiple species we can generalize the single-species BGK operator as $\mathcal{H}^s = \mathcal{H}^{ss} + \sum_{r\neq s}\mathcal{H}^{sr}$, where

$$\mathcal{H}^s[f^s] = \nu_{ss}\big(f^s_M - f^s\big); \quad \mathcal{H}^{sr}[f^s, f^r] = \nu_{sr}\big(f^{sr}_M - f^s\big) \tag{17}$$

where ν_{sr} is the collision-frequency of species s with species r and f^s_M, f^{sr}_M are a Maxwellians given by

$$f^s_M = n_s\left(\frac{m_s}{2\pi T^s}\right)^{3/2} e^{-m_s(\mathbf{v}-\mathbf{u}^s)^2/2T^s} \tag{18}$$

$$f^{sr}_M = n_s\left(\frac{m_s}{2\pi T^{sr}}\right)^{3/2} e^{-m_s(\mathbf{v}-\mathbf{u}^{sr})^2/2T^s} \tag{19}$$

and where $\mathbf{u}^{sr}$ and T^{sr}.

We can now compute the moments of the BGK operator and obtain the following

$$m_s\langle\mathcal{H}^s\rangle = 0 \tag{20}$$

$$m_s\langle\mathbf{v}\mathcal{H}^s\rangle = m_s n_s \sum_{r\neq s}\nu_{sr}\Delta\mathbf{u}^{sr} \tag{21}$$

$$m_s\langle c_i c_j\mathcal{H}^s\rangle = -\Pi^s_{ij}\big(\nu_{ss} + \sum_{r\neq s}\nu_{sr}\big) + n_s\delta_{ij}\sum_{r\neq s}\nu_{sr}\Delta T^{sr} + m_s n_s\sum_{r\neq s}\nu_{sr}\Delta u^{sr}_i\Delta u^{sr}_j \tag{22}$$

$$m_s\langle c_i c_j c_k\mathcal{H}^s\rangle = -Q^s_{ijk}\big(\nu_{ss} + \sum_{r\neq s}\nu_{sr}\big) + \sum_{r\neq s}\nu_{sr}\Delta u^{sr}_{[i}P^s_{jk]} + m_s n_s\sum_{r\neq s}\nu_{rs}\Delta u^{sr}_i\Delta u^{sr}_j\Delta u^{sr}_k \tag{23}$$

where we have defined $\Delta\mathbf{u}^{sr} = \mathbf{u}^{sr} - \mathbf{u}^s$ and $\Delta T^{sr} = T^{sr} - T^s$. Often we need the $m_s\langle v_i v_j\mathcal{H}^s\rangle$ as its trace is twice the total (internal plus kinetic) particle energy. From Eqns. (20)–(22) we can show that

$$m_s\langle v_i v_j\mathcal{H}^s\rangle = -\Pi^s_{ij}\big(\nu_{ss} + \sum_{r\neq s}\nu_{sr}\big) + n_s\delta_{ij}\sum_{r\neq s}\nu_{sr}\Delta T^{sr} + m_s n_s\sum_{r\neq s}\nu_{sr}\Delta u^{sr}_i\Delta u^{sr}_j + m_s n_s\sum_{r\neq s}\nu_{sr}u^s_{[i}\Delta u^{sr}_{j]}. \tag{24}$$

From this, the change in energy due to collisions can be computed by taking the trace:

$$2\Delta\mathcal{E}^s = 3n_s\sum_{r\neq s}\nu_{sr}\Delta T^{sr} + m_s n_s\sum_{r\neq s}\nu_{sr}(|\mathbf{u}^{sr}|^2 - |\mathbf{u}^s|^2) \tag{25}$$

For a two-species plasma, summing the electron and ion contribution, this leads to Eq. (7) in Greene[1].

4 Multi-species Lenard-Bernstein operator

Collisions in plasmas are not well approximated by BGK type operators as these approximate "hard-sphere" collisions. Instead, in a plasma, the collisions are all near grazing and need a Fokker-Planck type operator. A simple such operator is the Lenard-Bernstein operator (LBO) which, for multiple species, can be written as $\mathcal{H}^s = \mathcal{H}^{ss} + \sum_{r\neq s}\mathcal{H}^{sr}$ where

$$\mathcal{H}^{ss}[f^s] = \nu_{ss}\nabla_{\mathbf{v}}\cdot\Big((\mathbf{v}-\mathbf{u}^s)f^s + \frac{T^s}{m_s}\nabla_{\mathbf{v}}f^s\Big) \tag{26}$$

$$\mathcal{H}^{sr}[f^s, f^r] = \nu_{sr}\nabla_{\mathbf{v}}\cdot\Big((\mathbf{v}-\mathbf{u}^{sr})f^s + \frac{T^{sr}}{m_s}\nabla_{\mathbf{v}}f^s\Big) \tag{27}$$

where ν_{sr} is the collision-frequency of species s with species r, and $\mathbf{u}^{sr}$ and T^{sr} are (yet undetermined) drift speeds and temperatures. Note that in general, $\nu_{sr} \neq \nu_{rs}$. In fact, we need $n_s m_s \nu_{sr} = n_r m_r \nu_{rs}$, which is essentially a statement that a heavier particle is less affected by collisions than is a lighter particle. For computing moments it is more convenient to rewrite the cross-species collision term as follows

$$\mathcal{H}^{sr}[f^s, f^r] = \nu_{sr} \nabla_{\mathbf{v}} \cdot \big((\mathbf{v} - \mathbf{u}^s) f^s + \frac{T^s}{m_s} \nabla_{\mathbf{v}} f^s \big) - \nu_{sr} \Delta \mathbf{u}^{sr} \cdot \nabla_{\mathbf{v}} f^s + \nu_{sr} \frac{\Delta T^{sr}}{m_s} \nabla^2_{\mathbf{v}} f^s. \tag{28}$$

Note that the first term is now the same form as the self-collision term (except for the different collision frequency) with the other terms explicitly in the form of relaxation due to intespecies drag and diffusion.

We can now compute the moments of the LBO and obtain the following

$$m_s \langle \mathcal{H}^s \rangle = 0 \tag{29}$$

$$m_s \langle \mathbf{v} \mathcal{H}^s \rangle = m_s n_s \sum_{r \neq s} \nu_{sr} \Delta \mathbf{u}^{sr} \tag{30}$$

$$m_s \langle c_i c_j \mathcal{H}^s \rangle = -2 \Pi^s_{ij} \big(\nu_{ss} + \sum_{r \neq s} \nu_{sr} \big) + 2 n_s \delta_{ij} \sum_{r \neq s} \nu_{sr} \Delta T^{sr} \tag{31}$$

$$m_s \langle c_i c_j c_k \mathcal{H}^s \rangle = -3 Q^s_{ijk} \big(\nu_{ss} + \sum_{r \neq s} \nu_{sr} \big) + \sum_{r \neq s} \nu_{sr} \Delta u^{sr}_{[i} P^s_{jk]}. \tag{32}$$

Often we need $m_s \langle v_i v_j \mathcal{H}^s \rangle$ as its trace is twice the total (internal plus kinetic) particle energy. From Eqns. (29)–(31) we can show that

$$m_s \langle v_i v_j \mathcal{H}^s \rangle = -2 \Pi^s_{ij} \big(\nu_{ss} + \sum_{r \neq s} \nu_{sr} \big) + 2 n_s \delta_{ij} \sum_{r \neq s} \nu_{sr} \Delta T^{sr} + m_s n_s \sum_{r \neq s} \nu_{sr} u^s_{[i} \Delta u^{sr}_{j]}. \tag{33}$$

From this, the change in energy due to collisions can be computed by taking the trace:

$$2 \Delta \mathcal{E}^s = 6 n_s \sum_{r \neq s} \nu_{sr} \Delta T^{sr} + 2 m_s n_s \sum_{r \neq s} \nu_{sr} u^s_i \Delta u^{sr}_i. \tag{34}$$

Note that this is different than the BGK expression and will hence lead to different expressions than in Greene's paper for the intermediate velocities and temperatures.

References

[1] John M. Greene. Improved Bhatnagar-Gross-Krook model of electron-ion collisions. *The Physics of Fluids*, 16(11):2022–2023, 1973.