Solution to Maxwell Equations in Cylindrical Waveguides

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1 Basic Equations

In this document I derive solutions to Maxwell equations in (potentially coaxial) cylindrical waveguides. The frequency domain Maxwell equations are

$$-i\omega\mathbf{E} - \nabla\times\mathbf{B} = 0 \tag{1}$$
$$-i\omega\mathbf{B} + \nabla\times\mathbf{E} = 0. \tag{2}$$

Here we have assumed speed of light is one. Taking the curl of each equation and eliminating we can separate the equations for the electric and magnetic fields as[1]

$$-\omega^2\mathbf{E} + \nabla\times\nabla\times\mathbf{E} = 0 \tag{3}$$
$$-\omega^2\mathbf{B} + \nabla\times\nabla\times\mathbf{B} = 0. \tag{4}$$

As the free-space electric and magnetic fields are divergence free we can write these equations as

$$-\omega^2\mathbf{E} - \nabla^2\mathbf{E} = 0 \tag{5}$$
$$-\omega^2\mathbf{B} - \nabla^2\mathbf{B} = 0. \tag{6}$$

As the equations satisfied by the fields are the same, denote them by $\mathbf{F}$. In cylindrical coordinates (r, ϕ, z) we can write them in component form as

$$\omega^2 F_r + \nabla^2 F_r - \frac{2}{r^2}\frac{\partial F_\phi}{\partial\phi} - \frac{F_r}{r^2} = 0 \tag{7}$$
$$\omega^2 F_\phi + \nabla^2 F_\phi + \frac{2}{r^2}\frac{\partial F_r}{\partial\phi} - \frac{F_\phi}{r^2} = 0 \tag{8}$$
$$\omega^2 F_z + \nabla^2 F_z = 0, \tag{9}$$

where the Laplacian acting on a scalar is defined as

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial\phi^2} + \frac{\partial^2 f}{\partial z^2} = 0. \tag{10}$$

Let us examine the z component equation first. This can be written as

$$\omega^2 F_z + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial F_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 F_z}{\partial\phi^2} + \frac{\partial^2 F_z}{\partial z^2} = 0. \tag{11}$$

[1] To easily follow these derivations it is important to have the NRL Plasma Formulary at hand, specially the pages on vector identities.

We will assume solutions of the form $F_z(r,\phi,z) = F_z(r)e^{im\phi}e^{ik_n z}$, where $k_n = 2\pi n/L_z$, where L_z is the length of the cylinder. This leads to an ODE for F_z

$$r^2\frac{d^2F_z}{dr^2} + r\frac{dF_z}{dr} + \left[r^2(\omega^2 - k_n^2) - m^2\right]F_z = 0. \tag{12}$$

The solution to this equation can be written in terms of Bessel functions

$$F_z(r) = aJ_m\left[r\sqrt{\omega^2 - k_n^2}\right] + bY_m\left[r\sqrt{\omega^2 - k_n^2}\right] \tag{13}$$

where J_m and Y_m are Bessel functions of the first and second kind respectively, and a and b are constants.

2 Transverse Magnetic Mode

Now, consider applying this solution to determine the electric field inside a coaxial cylinder with inner and outer radii $r_0 > 0$ and $r_1 > r_0$. The boundary conditions for tangential electric field give

$$E_z(r_0) = E_z(r_1) = 0. \tag{14}$$

This leads to two equation

$$E_z(r_0) = aJ_m\left[r_0\sqrt{\omega^2 - k_n^2}\right] + bY_m\left[r_0\sqrt{\omega^2 - k_n^2}\right] = 0 \tag{15}$$

$$E_z(r_1) = aJ_m\left[r_1\sqrt{\omega^2 - k_n^2}\right] + bY_m\left[r_1\sqrt{\omega^2 - k_n^2}\right] = 0 \tag{16}$$

for the three unknowns a, b and ω. For non-trivial solutions this means that the frequency ω must be the roots of the equation

$$J_m\left[r_0\sqrt{\omega^2 - k_n^2}\right]Y_m\left[r_1\sqrt{\omega^2 - k_n^2}\right] - J_m\left[r_1\sqrt{\omega^2 - k_n^2}\right]Y_m\left[r_0\sqrt{\omega^2 - k_n^2}\right] = 0. \tag{17}$$

Once we find a root we can choose either one of a or b arbitrarily, and the other one is then determined from the above conditions.

Note that for a given value of k_n not all modes can propagate inside the coaxial cylinder. We must choose $\omega^2 - k_n^2 > 0$ if we want a propagating mode, otherwise the mode will damp. Note that in the special case in which we include the axis, i.e. $r_0 = 0$, we must set $b = 0$ as Y_m blows up at $r = 0$. Hence, in this case, the frequency is determined from the roots of

$$J_m\left[r_1\sqrt{\omega^2 - k_n^2}\right] = 0. \tag{18}$$

Once E_z is determined and assuming that $E_r = E_\phi = 0$, we can complete the solution by computing the magnetic field from Eq. (2). This gives

$$-i\omega B_r + \frac{1}{r}\frac{\partial E_z}{\partial \phi} = 0 \tag{19}$$

$$-i\omega B_\phi - \frac{\partial E_z}{\partial r} = 0. \tag{20}$$

In this solution, the magnetic field is perpendicular to the direction of propagation (Z-direction), and hence the mode is called *transverse magnetic mode*.

3 Transverse Electric Mode

Now consider we have $B_\phi = B_r = 0$ and B_z is determined as $B_z(r)e^{im\phi}e^{ik_n z}$, with $B_z(r)$ computed as a linear combination of Bessel functions as above. Once we have B_z we can calculate the transverse electric field as

$$-i\omega E_r - \frac{1}{r}\frac{\partial B_z}{\partial \phi} = 0 \tag{21}$$

$$-i\omega E_\phi + \frac{\partial B_z}{\partial r} = 0. \tag{22}$$

The boundary conditions now imply that $E_\phi(r_0) = E_\phi(r_1) = 0$. Hence, as above, we can derive that the frequencies in this case are the roots of the equation

$$J'_m\big[r_0\sqrt{\omega^2 - k_n^2}\big]Y'_m\big[r_1\sqrt{\omega^2 - k_n^2}\big] - J'_m\big[r_1\sqrt{\omega^2 - k_n^2}\big]Y'_m\big[r_0\sqrt{\omega^2 - k_n^2}\big] = 0 \tag{23}$$

where primes denote derivatives. Once we find a root, the solution is then completed by selecting either a or b and using the boundary condition to determine the other. In this solution, the electric field is perpendicular to the direction of propagation (Z-direction), and hence the mode is called *transverse electric mode*.