Moment equations and notes on linear dispersion solvers in `Gkeyll`

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1 Generic moment equations

Each species in a multi-component plasma is described by the Boltzmann equation which describes the temporal evolution of the particle distribution function in a six dimensional spatial and velocity space and evolves under the influence of collisions and electromagnetic forces. With the distribution function for species s, $f^s(\mathbf{x}, \mathbf{v}, t)$, defined such that $f^s(\mathbf{x}, \mathbf{v}, t) d\mathbf{x} d\mathbf{v}$ is the number of particles in a phase-space volume element $d\mathbf{x} d\mathbf{v}$, the Boltzmann equation may be written as

$$\frac{\partial f^s}{\partial t} + v_j \frac{\partial f^s}{\partial x_j} + \frac{q_s}{m_s}(E_j + \epsilon_{kmj} v_k B_m) \frac{\partial f^s}{\partial v_j} = \mathcal{H}^s \tag{1}$$

Here $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic flux density, q_s and m_s are the charge and mass of the plasma species and ϵ_{kmj} is the completely anti-symmetric pseudo-tensor which is defined to be ± 1 for even/odd permutations of $(1, 2, 3)$ and zero otherwise. Summation over repeated indices is assumed. Of course, the EM fields are determined from Maxwell equations

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0 \tag{2}$$

$$\epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} = -\mu_0 \mathbf{J} \tag{3}$$

where the current $\mathbf{J}$ is computed from moments of the distribution function

$$\mathbf{J} = \sum_s \int_{-\infty}^{\infty} \mathbf{v} f^s \, d^3\mathbf{v}. \tag{4}$$

General moment equations can be derived from the Boltzmann equation. To do this we adopt the following definition for the moments

$$P^{(n)}_{i_1 i_2 \ldots i_n} \equiv m \int_{-\infty}^{\infty} c_{i_1} c_{i_2} \ldots c_{i_n} f d^3\mathbf{v} \tag{5}$$

and generalized friction from collisions

$$R^{(n)}_{i_1 i_2 \ldots i_n} \equiv m \int_{-\infty}^{\infty} c_{i_1} c_{i_2} \ldots c_{i_n} \mathcal{H} d^3\mathbf{v} \tag{6}$$

Here, the species index is dropped and $\mathbf{c} \equiv \mathbf{v} - \mathbf{u}$. With this definition, for example, $P^{(0)} = mn(\mathbf{x}, t)$, where $n(\mathbf{x}, t)$ is the number density, $P^{(1)}_i = 0$, $P^{(2)}_{ij} = P_{ij}$, where P_{ij} is the pressure tensor and $P^{(3)}_{ijk} = Q_{ijk}$, where Q_{ijk} is the heat flux tensor, etc.

The non-conservative form of the general moment equations are as follows. For $n = 1$ we have

$$\partial_t u_{i_1} + u_j \partial_j u_{i_1} + \frac{1}{P^{(0)}} \partial_j P^{(2)}_{ji_1} = \frac{q}{m}(E_{i_1} + B_m u_k \epsilon_{kmi_1}) + \frac{1}{P^{(0)}} R^{(1)}_{i_1}. \tag{7}$$

For $n \neq 1$ (including $n = 0$) we have

$$\begin{aligned}\partial_t P^{(n)}_{i_1 i_2 \ldots i_n} - \frac{1}{P^{(0)}} \partial_j P^{(2)}_{j[i_1} P^{(n-1)}_{i_2 \ldots i_n]} + \partial_j P^{(n+1)}_{j i_1 i_2 \ldots i_n} + \partial_j u_j P^{(n)}_{i_1 i_2 \ldots i_n} + \partial_j u_{[i_1} P^{(n)}_{i_2 \ldots i_n]j} + u_j \partial_j P^{(n)}_{i_1 i_2 \ldots i_n} \\ = \frac{q}{m} B_m \epsilon_{km[i_1} P^{(n)}_{i_2 \ldots i_n]k} + R^{(n)}_{i_1 i_2 \ldots i_n} - \frac{1}{P^{(0)}} R^{(1)}_{[i_1} P^{(n-1)}_{i_2 \ldots i_n]}\end{aligned} \tag{8}$$

In these equations square brackets around indices represent the minimal sum over permutations of free indices needed to yield completely symmetric tensors[1].

The general conservative form of the moment equations can be derived. For this the following total velocity moments are defined

$$\mathcal{P}^{(n)}_{i_1 i_2 \ldots i_n} \equiv m \int_{-\infty}^{\infty} v_{i_1} v_{i_2} \ldots v_{i_n} f d^3\mathbf{v} \tag{9}$$

and generalized total velocity friction from collisions

$$\mathcal{R}^{(n)}_{i_1 i_2 \ldots i_n} \equiv m \int_{-\infty}^{\infty} v_{i_1} v_{i_2} \ldots v_{i_n} \mathcal{H} d^3\mathbf{v} \tag{10}$$

With these the general conservative form of the equations are

$$\partial_t \mathcal{P}^{(n)}_{i_1 \ldots i_n} + \partial_j \mathcal{P}^{(n+1)}_{j i_1 \ldots i_n} = \frac{q}{m}(E_{[i_1} \mathcal{P}^{(n-1)}_{i_2 \ldots i_n]} + B_m \epsilon_{mj[i_1} \mathcal{P}^{(n)}_{i_2 \ldots i_n j]}) + \mathcal{R}^{(n)}_{i_1 \ldots i_n}. \tag{11}$$

2 Ten- and five-moment equations

We are mostly interested in the equations for the lower moments, in particular, the number density, the velocity and the pressure tensor. Specializing the general equations we get

$$\partial_t n + u_j \partial_j n + n \partial_j u_j = 0 \tag{12}$$

$$\partial_t u_i + u_j \partial_j u_i + \frac{1}{mn} \partial_j P_{ji} = \frac{q}{m}(E_i + \epsilon_{kmi} u_k B_m) \tag{13}$$

$$\partial_t P_{mn} + \partial_j Q_{jmn} + \partial_j u_j P_{mn} + \partial_j u_{[m} P_{n]j} + u_j \partial_j P_{mn} = \frac{q}{m} B_r \epsilon_{kr[m} P_{n]k}. \tag{14}$$

This is a system of ten equations for each plasma species. To close this system we need a relation to determine the heat-flux tensor Q_{ijk}. For now we will leave this unspecified.

Now write the pressure tensor as a scalar and trace-free part

$$P_{ij} = p\delta_{ij} + \Pi_{ij} \tag{15}$$

[1]For example, $u_{[i}E_{j]} = u_i E_j + u_j E_i$. In general, as the moments are themselves symmetric, one usually only needs to cyclically permute the free indices and sum to get a symmetric expression.

where $p = P_{ii}/3$ and $\Pi_{ii} = 0$. Substitute this in the above equation and also take the trace of the pressure-tensor equation to get the five-moment equations

$$\partial_t n + u_j \partial_j n + n\partial_j u_j = 0 \tag{16}$$

$$\partial_t u_i + u_j \partial_j u_i + \frac{1}{mn}\left(\partial_i p + \partial_j \Pi_{ji}\right) = \frac{q}{m}(E_i + \epsilon_{kmi} u_k B_m) \tag{17}$$

$$\partial_t p + u_j \partial_j p + \frac{5}{3} p\, \partial_j u_j + \frac{2}{3}\Pi_{mj}\partial_j u_m + \frac{2}{3}\partial_j q_j = 0. \tag{18}$$

This is a system of five equations for each plasma species. To close this system of equation we need to specify a closure for Π_{ij} and the heat-flux vector $q_j \equiv Q_{jmm}/2$. For now, we will leave these unspecified.

A further simplification can be made to the five-moment equations by assuming that the fluid is *isothermal*. With this we can drop the pressure equation and assume the temperature is constant. Note that the temperature can be further set to zero, i.e. the fluid assumed to be cold.

3 An eigenvalue approach to linear dispersion solvers

Consider a generic system of equations

$$\partial_t \mathbf{Q} + \mathbf{A}_j \partial_j \mathbf{Q} = \mathbf{S}(\mathbf{Q}) \tag{19}$$

where $\mathbf{Q}(\mathbf{x}, t)$ is a vector of variables, $\mathbf{A}_j(\mathbf{Q})$ are matrices and $\mathbf{S}(\mathbf{Q})$ is the source vector. We will linearize this around a uniform equilibrium $\mathbf{Q}_0$ and write $\mathbf{Q} = \mathbf{Q}_0 + \mathbf{Q}_1$ to get the linear system

$$\partial_t \mathbf{Q}_1 + \mathbf{A}_j(\mathbf{Q}_0)\partial_j \mathbf{Q}_1 = \mathbf{M}(\mathbf{Q}_0)\mathbf{Q}_1 \tag{20}$$

where $\mathbf{M}(\mathbf{Q}) \equiv \partial \mathbf{S}/\partial \mathbf{Q}$ is the Jacobian of the source terms. Computing the Fourier transform of this in the standard way we get

$$-i\omega \mathbf{Q}_1 + ik_j \mathbf{A}_j(\mathbf{Q}_0) = \mathbf{M}(\mathbf{Q}_0)\mathbf{Q}_1. \tag{21}$$

Rearranging this we get

$$[k_j \mathbf{A}_j(\mathbf{Q}_0) + i\mathbf{M}(\mathbf{Q}_0) - \omega \mathbf{I}]\mathbf{Q}_1 = 0. \tag{22}$$

Hence for non-trivial solutions to the linearized system we need the frequencies $\omega(\mathbf{k})$ to be the eigenvalues of the matrix

$$\mathbf{D}(\mathbf{k}, \mathbf{Q}_0) \equiv k_j \mathbf{A}_j(\mathbf{Q}_0) + i\mathbf{M}(\mathbf{Q}_0). \tag{23}$$

For various closures and also for the case when we have an electrostatic problem we need to add additional terms to the $\mathbf{D}$ matrix, but the essential idea of constructing an eigenvalue problem remains the same.

4 The electrostatic limit

For electrostatic problems we have

$$\mathbf{E} = -\nabla \phi \tag{24}$$

where ϕ is the electrostatic potential which is determined from Poisson equation

$$\nabla^2 \phi = -\frac{1}{\epsilon_0} \sum_s q_s n_s. \tag{25}$$

Hence, in k-space we can eliminate the electric field from the linearized equations by writing

$$\mathbf{E} = -\frac{i\mathbf{k}}{k^2 \epsilon_0} \sum_s q_s n_s \tag{26}$$

where now n_s is the perturbed number density. Note that this has the effect of coupling all species to each other via the momentum equation through their dependence on the perturbed density of all other species.