Mirror force, drifts and a simplified gyrokinetic/fluid model in 1D

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1 The Momentum Equation

Start from the fluid momentum equation

$$\frac{d\mathbf{u}_s}{dt} + \frac{1}{m_s n_s}\nabla\cdot\mathbf{P}_s = \frac{q_s}{m_s}(\mathbf{E} + \mathbf{u}_s\times\mathbf{B}), \tag{1}$$

where

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{u}_s\cdot\nabla \tag{2}$$

and all other symbols have their usual meaning. Note this equation is exact, i.e. this is the exact first moment, $\langle \mathbf{v} f_s\rangle$, of the Vlasov equation.

Now, consider a magnetized plasma, and split the pressure tensor as

$$\mathbf{P}_s = \mathbf{P}_s^C + \mathbf{\Pi}_s \tag{3}$$

where

$$\mathbf{P}_s^C = (\mathbf{I} - \mathbf{bb})p_{s,\perp} + \mathbf{bb}p_{s,\|} = \mathbf{I}p_{s,\perp} + \mathbf{bb}(p_{s,\|} - p_{s,\perp}) \tag{4}$$

is the CGL pressure tensor, $\mathbf{b} = \mathbf{B}/B$ is the direction of the magnetic field, and $\mathbf{\Pi}_s$ is the agyrotropic part of the pressure tensor. Note that $\mathrm{Tr}\,(\mathbf{P}_s^C) = 2p_{s,\perp} + p_{s,\|} = 3p_s$, where p_s is the scalar pressure. This also shows that $\mathrm{Tr}\,(\mathbf{\Pi}_s) = 0$.

The divergence of the CGL pressure tensor is

$$\nabla\cdot\mathbf{P}_s^C = \nabla p_{s,\perp} + (p_{s,\|} - p_{s,\perp})\underbrace{\nabla\cdot(\mathbf{bb})}_{(\nabla\cdot\mathbf{b})\mathbf{b} + \nabla_\|\mathbf{b}} + \mathbf{b}\nabla_\|(p_{s,\|} - p_{s,\perp}) \tag{5}$$

where $\nabla_\| \equiv \mathbf{b}\cdot\nabla$.

1.1 Parallel component: the mirror force

Now consider the parallel component of $\nabla\cdot\mathbf{P}_s^C$:

$$\mathbf{b}\cdot(\nabla\cdot\mathbf{P}_s^C) = \nabla_\| p_{s,\perp} + (p_{s,\|} - p_{s,\perp})\big(\nabla\cdot\mathbf{b} + \mathbf{b}\cdot(\nabla_\|\mathbf{b})\big) + \nabla_\|(p_{s,\|} - p_{s,\perp}) \tag{6}$$

Now $\mathbf{b}\cdot(\nabla_\|\mathbf{b}) = \nabla_\|(\mathbf{b}\cdot\mathbf{b})/2 = 0$. Also

$$\nabla\cdot\mathbf{b} = \nabla\cdot\frac{\mathbf{B}}{B} = \frac{1}{B}\nabla\cdot\mathbf{B} + \mathbf{B}\cdot\nabla\frac{1}{B} = -\frac{1}{B}\nabla_\| B. \tag{7}$$

Putting everything together we finally get the parallel momentum equation

$$\mathbf{b}\cdot\frac{d\mathbf{u}_s}{dt}=\frac{q_s}{m_s}E_{\|}-\frac{1}{n_s m_s}\nabla_{\|}p_{s,\|}+\frac{(p_{s,\|}-p_{s,\perp})}{n_s m_s B}\nabla_{\|}B+\mathbf{b}\cdot(\nabla\cdot\mathbf{\Pi}_s)\,. \tag{8}$$

Now, from single-particle orbit theory we can show the mirror force for a single particle is $-\mu_s\nabla_{\|}B$, where μ_s is the magnetic moment. Hence, integrating over the particle distribution function we get the net mirror force is

$$-\nabla_{\|}B\int_{-\infty}^{\infty}\frac{m_s v_\perp^2}{2B}f_s\,d^2\mathbf{v}=-\frac{p_{s,\perp}}{B}\nabla_{\|}B. \tag{9}$$

Hence, in the parallel momentum equation, only the term with $p_{s,\perp}$ is due to the mirror force.

For isotropic pressure tensor ($p_{s,\|}=p_{s,\perp}=p_s$) the parallel momentum reduces to

$$\mathbf{b}\cdot\frac{d\mathbf{u}_s}{dt}=\frac{q_s}{m_s}E_{\|}-\frac{1}{n_s m_s}\nabla_{\|}p_s. \tag{10}$$

It may appear that the mirror force has vanished. However, that is not the case. When the pressure tensor is isotropic, the mirror force exactly balances the pressure gradient from the change in area of a flux tube.

1.2 Perpendicular component: drifts across magnetic field

The general procedure to derive drifts is to look at the motion of the plasma perpendicular to the magnetic field. To do this, take the cross product of the momentum equation with $\mathbf{B}$ to get

$$\frac{d\mathbf{u}_s}{dt}\times\mathbf{B}+\frac{1}{m_s n_s}\nabla\cdot\mathbf{P}_s\times\mathbf{B}=\frac{q_s}{m_s}\mathbf{E}\times\mathbf{B}+\frac{q_s}{m_s}\underbrace{(\mathbf{u}_s\times\mathbf{B})\times\mathbf{B}}_{(\mathbf{B}\cdot\mathbf{u}_s)\mathbf{B}-B^2\mathbf{u}_s} \tag{11}$$

Rearranging, the perpendicular component of the fluid velocity is

$$\mathbf{u}_{s\perp}=\frac{\mathbf{E}\times\mathbf{B}}{B^2}-\frac{\nabla\cdot\mathbf{P}_s\times\mathbf{B}}{q_s n_s B^2}-\frac{m_s}{q_s B^2}\frac{d\mathbf{u}_s}{dt}\times\mathbf{B}. \tag{12}$$

we can calculate the contribution of the CGL pressure tensor to the diamagnetic drifts as

$$-\frac{\nabla\cdot\mathbf{P}_s^C\times\mathbf{B}}{q_s n_s B^2}=-\frac{\nabla p_{s,\perp}\times\mathbf{B}}{q_s n_s B^2}+(p_{s,\perp}-p_{s,\|})\frac{\nabla_{\|}\mathbf{b}\times\mathbf{B}}{q_s n_s B^2} \tag{13}$$

Putting everything together we get

$$\mathbf{u}_{s\perp}=\frac{\mathbf{E}\times\mathbf{B}}{B^2}-\frac{\nabla p_{s,\perp}\times\mathbf{B}}{q_s n_s B^2}+(p_{s,\perp}-p_{s,\|})\frac{\nabla_{\|}\mathbf{b}\times\mathbf{B}}{q_s n_s B^2}-\frac{\nabla\cdot\mathbf{\Pi}_s\times\mathbf{B}}{q_s n_s B^2}-\frac{m_s}{q_s B^2}\frac{d\mathbf{u}_s}{dt}\times\mathbf{B}. \tag{14}$$

Now, define a magnetization vector for each species as $\nabla\times\mathbf{M}_s$ where

$$\mathbf{M}_s=-\mathbf{b}\int_{-\infty}^{\infty}\frac{m_s v_\perp^2}{2B}f_s\,d\mathbf{v}^3=-p_{s,\perp}\frac{\mathbf{B}}{B^2} \tag{15}$$

Hence, we have

$$\nabla\times\mathbf{M}_s=\nabla\times\left(-p_{s,\perp}\frac{\mathbf{B}}{B^2}\right)=-\frac{\nabla p_{s,\perp}\times\mathbf{B}}{B^2}-p_{s,\perp}\nabla\times\left(\frac{\mathbf{b}}{B}\right). \tag{16}$$

Substituting in Eq. (14) we finally get

$$\mathbf{u}_{s\perp} = \frac{\mathbf{E}\times\mathbf{B}}{B^2} + \frac{\nabla\times(-p_{s,\perp}\mathbf{b}/B)}{q_s n_s} + \frac{p_{s,\perp}\nabla\times(\mathbf{b}/B)_\perp}{q_s n_s} + (p_{s,\perp}-p_{s,\parallel})\frac{\nabla_\parallel \mathbf{b}\times\mathbf{B}}{q_s n_s B^2} - \frac{\nabla\cdot\mathbf{\Pi}_s\times\mathbf{B}}{q_s n_s B^2} - \frac{m_s}{q_s B^2}\frac{d\mathbf{u}_s}{dt}\times\mathbf{B}. \tag{17}$$

The first term is the $\mathbf{E}\times\mathbf{B}$ drift, the second term, the diamagnetic drift, the third and fourth terms the gradient and curvature drifts, while the last term contain the polarization drifts.

In a typical mirror geometry the drifts are small compared to the parallel momentum terms. However, in the spinning mirror, of course, the $\mathbf{E}\times\mathbf{B}$ velocity needs to be accounted for as it provides the rotation to the plasma. Also, the drift terms can lead to instabilities.

2 A Gyrokinetic Viewpoint

In a simplified model that follows a field-line, the gyrokinetic equation can be written in the form

$$\frac{\partial}{\partial t}(Bf_s) + B\frac{\partial}{\partial z}\left(\frac{1}{B}\dot{z}Bf_s\right) + \frac{\partial}{\partial v_\parallel}(\dot{v}_\parallel B f_s) = 0 \tag{18}$$

where $f_s(z, v_\parallel, \mu)$ is the distribution function of the gyrocenters, z is the length along the field-line. The factor of $1/B$ comes about from the Jacobian of the transform to field-line following coordinates. The characteristic velocities are given by

$$\dot{z} = v_\parallel \tag{19}$$

$$\dot{v}_\parallel = -\frac{1}{m_s}\left(\mu\frac{\partial B}{\partial z} + q_s\frac{\partial\phi}{\partial z}\right) \tag{20}$$

Note that the mirror force explicitly appears in the parallel acceleration. We can take the velocity moment of this equation to derive the gyrokinetic parallel momentum equation. To do this, the moments need to be computed carefully: the volume-element in phase-space is

$$d^3\mathbf{v} = B\,d\mathbf{w} = B\,2\pi m_s^{-1}\,dv_\parallel d\mu \tag{21}$$

and the limits of integration $v_\parallel \in [-\infty, \infty]$ and $\mu \in [0, \infty]$. (Note that the volume-element definition comes about from the definition of the magnetic moment). For example, the moments we will use below are defined as

$$n_s = \int f_s B\,d\mathbf{w} \tag{22}$$

$$n_s u_{s,\parallel} = \int v_\parallel f_s B\,d\mathbf{w} \tag{23}$$

$$p_{s,\parallel} = m_s\int (v_\parallel - u_{s,\parallel})^2 f_s B\,d\mathbf{w} \tag{24}$$

$$p_{s,\perp} = B\int \mu f_s B\,d\mathbf{w} \tag{25}$$

$$q_{s,\parallel} = m\int (v_\parallel - u_{s,\parallel})^3 f_s B\,d\mathbf{w} \tag{26}$$

$$q_{s,\perp} = B\int (v_\parallel - u_{s,\parallel})\mu B f\,d\mathbf{w}. \tag{27}$$

First, we can derive the number density equation by integrating the gyrokinetic equation over all velocity space to get

$$\frac{\partial n_s}{\partial t} + B\frac{\partial}{\partial z}\left(\frac{n_s u_{s,\|}}{B}\right) = 0. \tag{28}$$

Next, take the $m_s v_\|$ moment of the GK equation to get

$$\frac{\partial}{\partial t}(m_s n_s u_{s,\|}) + B\frac{\partial}{\partial z}\left[\frac{1}{B}(p_{s,\|} + m_s n_s u_{s,\|}^2)\right] = -\frac{p_{s,\perp}}{B}\frac{\partial B}{\partial z} - \frac{q_s n_s}{m_s}\frac{\partial \phi}{\partial z}. \tag{29}$$

Rearrange to get

$$\frac{\partial}{\partial t}(m_s n_s u_{s,\|}) + B\frac{\partial}{\partial z}\left(\frac{m_s n_s u_{s,\|}^2}{B}\right) = -q_s n_s\frac{\partial \phi}{\partial z} - \frac{\partial p_{s,\|}}{\partial z} + \frac{(p_{s,\|} - p_{s,\perp})}{B}\frac{\partial B}{\partial z}. \tag{30}$$

Using the continuity equation in this we can finally write

$$\frac{\partial u_{s,\|}}{\partial t} + u_{s,\|}\frac{\partial u_{s,\|}}{\partial z} = -\frac{q_s}{m_s}\frac{\partial \phi}{\partial z} - \frac{1}{m_s n_s}\frac{\partial p_{s,\|}}{\partial z} + \frac{(p_{s,\|} - p_{s,\perp})}{m_s n_s B}\frac{\partial B}{\partial z}. \tag{31}$$

The right-hand size of this expression is identical to Eq. (8), except for the agyrotropic terms are missing, consistent with the gyrokinetic approximation.

Instead of equations for pressure evolution, we will derive equations for particle energy for each species, $\mathcal{E}_s$:

$$\mathcal{E}_s \equiv \int \left(\frac{1}{2}m_s v_\|^2 + \mu B\right) f_s B \, d\mathbf{w} = \frac{1}{2}m_s u_{s,\|}^2 + \underbrace{\frac{p_{s,\|} + 2p_{s,\perp}}{2}}_{3p_s/2} \tag{32}$$

where p_s is the scalar pressure. This leads to

$$\frac{\partial \mathcal{E}_s}{\partial t} + B\frac{\partial}{\partial z}\left[\frac{(\mathcal{E}_s + p_{s,\|})u_{s,\|}}{B}\right] + B\frac{\partial}{\partial z}\left[\frac{q_{s,\|}/2 + q_{s,\perp}}{B}\right] = -\frac{q_{s,\perp} + u_{s,\|}p_{s,\perp}}{B}\frac{\partial B}{\partial z} - q_s n_s u_{s,\|}\frac{\partial \phi}{\partial z}. \tag{33}$$

The second term above is the advective energy flux, the third the energy flux due to non-ideal (heat-flux) effects and the RHS terms represent the energy exchange with other species and the field. We still need one more equation to determine $p_{s,\perp}$ (say) which we can derive by taking the μB moment to get

$$\frac{\partial p_{s,\perp}}{\partial t} + B\frac{\partial}{\partial z}\left(\frac{q_{s,\perp} + u_{s,\|}p_{s,\perp}}{B}\right) = 0. \tag{34}$$

Hence, the final set of equations we need to solve are the continuity equation Eq. (28), the momentum density equation Eq. (29), the particle energy equation Eq. (33) and the equation for perpendicular pressure Eq. (34). Of course, we still need a closure term to determine the parallel and perpendicular heat-fluxes. We may also need to add some collisions to account for pitch-angle scattering and inter-species energy equilibration.