Indexing operators for dimensionally independent grid indexing

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1 Notation for dimensionally independent grid indexing

To allow dimensionally independent grid indexing we will introduce the following operators (see Fig. 1)

- Δ_p and Δ_m work on cell indices and shift to the right and left cell index respectively.
- d_p and d_m work on edge indices and shift it to the right and left full index respectively.
- e_p and e_m work on cell indices and shift it to the right and left edge index respectively.

Let I be the unity operator, that is, an operator which keeps the index unchanged. Clearly, we have the relations, for example $\Delta_p\Delta_m = \Delta_m\Delta_p = I$, $d_p e_m = d_m e_p = I$, $e_p d_p = \Delta_p$ and $e_m d_m = \Delta_m$. Operator combinations allow moving to cells right of right of a given cell, for example: $d_p d_p$ or or $\Delta_p\Delta_p$. We will write such combinations as d_{2p} and Δ_{2p}. Also note that, for example, $\Delta_p f \Delta_p g = \Delta_p(fg)$ etc.

We will next define additional *transverse* shift-operators T_p and T_m that increment indices in the transverse direction. Optionally, we will attach a superscript to these operator (so for example T_p^1 or T_p^2) if there is more than one transverse direction.

Next we introduce the *directional modifiers* X, Y and Z. These determine the meaning of "right", "left" and "transverse". Modifers will always appear on the *left* of the various shift operators. For example $X\Delta_p$ means the index pair (i, j) becomes $(i + 1, j)$ while $Y\Delta_p$ means the index pair (i, j) becomes $(i, j + 1)$. Also note that modifiers acting on unity operator produce the unity operator. So $XI = YI = ZI = I$.

We will also use the *cyclic permutation* rule of determing transverse direction. So in 2D XT_p will shift (i, j) to $(i, j + 1)$ while YT_p will shift (i, j) to $(i + 1, j)$. However, in 3D YT_p will shift (i, j, k) to $(i, j, k + 1)$.

Finally we must evaluate *operator composition first* before applying to an index. That is when we write Xd_pf we actually mean $(Xd_p)f$. Also, that for 1D problems we will always drop directional modifier and implicitly assume unspecified modifier is X.

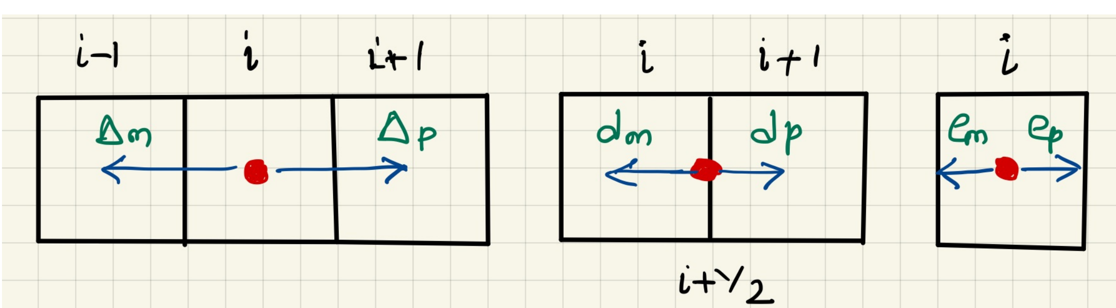

Figure 1: Basic indexing operators to move from cell to cell, face to cell and cell to face. These can be combined with each other, with transverse operators and directional modifiers to express stencils in a compact and dimensionally independent manner.

All this may appear very confusing. However, a couple of examples will suffice. Consider the 1D diffusion equation

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} \tag{1}$$

Using these operators (recall missing directional modifier is X) the central difference stencil becomes

$$\frac{\partial f_i}{\partial t} = \frac{1}{\Delta x^2}\left(f_{i+1} - 2f_i + f_{i-1}\right) = \frac{1}{\Delta x^2}\left(\Delta_p - 2I + \Delta_m\right) f_i. \tag{2}$$

As a slightly more complicated example in 3D, the stencil for the Laplacian $\nabla^2 f$ on a square grid with $\Delta x = \Delta y = \Delta z = h$ can be written as

$$\nabla^2 f = \frac{1}{h^2}(X + Y + Z)\left(\Delta_p - 2I + \Delta_m\right) f \tag{3}$$

Note that in this expression we have assumed $f = f_{i,j,k}$ and dropped the subscripts. Sometimes it is useful to retain these subscripts but often they can simply be dropped and implicitly clear from the context of the operators. We can factor this expression using some of the identities listed above to

$$\nabla^2 f = \frac{1}{h^2}\left[(X + Y + Z)\left(\Delta_p - \Delta_m\right) - \frac{6I}{h^2}\right] f. \tag{4}$$

After a little practice one can write down stencils almost "by inspection" quickly in a dimensionally independent manner.

2 Properties of indexing operators

We can easily derive the following identities that are often useful while proving properties of schemes.

$$\sum_{i=1}^{N} \phi_i \Delta_p f_i = \sum_{i=1}^{N} \Delta_m \phi_i f_i + \phi_N f_{N+1} - \phi_0 f_1 \tag{5}$$

$$\sum_{i=1}^{N} \phi_i \Delta_m f_i = \sum_{i=1}^{N} \Delta_p \phi_i f_i + \phi_1 f_0 - \phi_{N+1} f_N. \tag{6}$$