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Fokker-Planck Equation for an Inverse-Square Force*

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The contribution to the Fokker-Planck equation for the distribution function for gases, due to particle-particle interactions in which the fundamental two-body force obeys an inverse square law, is investigated. The coefficients in the equation, $\langle\Delta\mathbf{v}\rangle$ (the average change in velocity in a short time) and $\langle\Delta\mathbf{v}\Delta\mathbf{v}\rangle$, are obtained in terms of two fundamental integrals which are dependent on the distribution function itself. The transformation of the equation to polar coordinates in a case of axial symmetry is carried out. By expanding the distribution function in Legendre functions of the angle, the equation is cast into the form of an infinite set of one-dimensional coupled nonlinear integro-differential equations. If the distribution function is approximated by a finite series, the resultant Fokker-Planck equations may be treated numerically using a computing machine. Keeping only one or two terms in the series corresponds to the approximations of Chandrasekhar, and Cohen, Spitzer and McRoutly, respectively.

I. INTRODUCTION

IN dealing with the nonequilibrium properties of systems whose particles obey an inverse-square law of interaction, it is convenient to make use of the fact that under most circumstances small-angle collisions are much more important than collisions resulting in large momentum changes.[1] This leads to the method often used for treating such systems, in which one expands the collision integrand of the Boltzmann equation in powers of the momentum change per collision.

A more generally valid approach to the problem of treating changes in a distribution function resulting from frequently occurring "events," each of which produces a small change in the momentum of a particle, is to use the Fokker-Planck equation, which has been discussed by Chandrasekhar.[2] He has used the formalism of this equation to evaluate the collision terms of the Boltzmann equation under the assumptions that (a) the events producing changes in particle momenta are two-body interactions described by the associated differential scattering cross sections, and (b) that the distribution function is isotropic in space and velocity space. Spitzer and collaborators[3,4] have extended this calculation to the case in which the distribution function is of the form $f^{(0)}+\mu f^{(1)}$, where $f^{(0)}$ and $f^{(1)}$ are isotropic and μ is the direction cosine between the particle trajectory and some preferred direction in space, and $f^{(1)}$ is assumed to be small.

It is the purpose of this paper to present the mechanics of two-body collisions in a somewhat simplified form, and to derive the Fokker-Planck equation for an arbitrary distribution function. From this general equation such special cases as those of Chandrasekhar and Spitzer may easily be obtained. For more complex situations the equation is suitable for integration by an electronic computer.

II. FORMULATION OF THE PROBLEM

The Boltzmann equation for the change of the molecular distribution function is given by

$$\frac{\partial f_a}{\partial t}+v^\mu\frac{\partial f_a}{\partial x^\mu}+\frac{F^\mu}{m}\frac{\partial f_a}{\partial v^\mu}=\left(\frac{\partial f_a}{\partial t}\right)_c, \tag{1}$$

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[1] S. Chapman and T. G. Cowling, *Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, London, 1952), second edition, pp. 178–179.

[2] S. Chandrasekhar, Revs. Modern Phys. **15**, 1 (1943).

[3] Cohen, Spitzer, and McRoutly, Phys. Rev. **80**, 230 (1950). A more complete list of references is given in this paper.

[4] L. Spitzer and R. Harm, Phys. Rev. **89**, 977 (1953).

where f_a is the number of molecules of type a per unit volume in the phase space of configuration and velocity, and $\mathbf{F}$ is an external force field. The summation on repeated Greek indices is understood in this paper, while sums on Roman letters denoting molecular species are explicitly indicated. The quantity $(\partial f_a/\partial t)_c$ represents the change in the distribution function produced by collisions, and it is this term with which we are concerned. Since the interactions take place between molecules within the same small region in space, we need only consider the velocity dependence of the distribution function in evaluating this term.

The Fokker-Planck equation, which is simply a conservation equation, gives the time rate of change of f due to collisions as

$$\left(\frac{\partial f_a}{\partial t}\right)_c=-\frac{\partial}{\partial v^\mu}(f_a\langle\Delta v^\mu\rangle_a)+\frac{1}{2}\frac{\partial^2}{\partial v^\mu\partial v^\nu}(f_a\langle\Delta v^\mu\Delta v^\nu\rangle_a),\quad(2)$$

where v^μ is the component of particle velocity in Cartesian coordinates and $\langle\Delta v^\mu\rangle_a$ is the average increment per unit time of the μth component of velocity of a molecule of type a. The derivation of this equation rests on the approximation that small changes in v^μ are the most probable and that terms involving higher powers of Δv^μ contribute negligibly to $(\partial f_a/\partial t)_c$.[2] In the next section we give a more precise statement of the approximation made here.

In calculating the average values $\langle\Delta v^\mu\rangle$ and $\langle\Delta v^\mu\Delta v^\nu\rangle$, we make the usual assumption that changes in velocity v^μ result from two-particle interactions, or collisions during which spatial correlation effects (polarization effects or multiple collisions) are unimportant. For many situations this assumption is believed to be justified, as is indicated by the work of Chapman, Ferraro, and Persico,[1] and more recently, Gasiorowicz, Neuman, and Riddell.[5] The expression for $\langle\Delta v^\mu\rangle_a$ becomes

$$\langle\Delta v^\mu\rangle_a=\sum_b\int d\mathbf{v}'f_b(v'^\mu)\int d\Omega\sigma(u,\Omega)u\Delta v^\mu,\quad(3)$$

where u is the magnitude of the relative velocity $|v'^\mu-v^\mu|$, $\sigma(u)$ is the differential scattering cross section, Ω is the scattering solid angle, and Δv^μ is the change in v'^μ resulting from the collision. The increment Δv^μ has been integrated over all scattering angles, all velocities v^μ of the scattering particle, and has been summed over all the species of particles. Similarly the average value $\langle\Delta v^\mu\Delta v^\nu\rangle_a$ is given by

$$\langle\Delta v^\mu\Delta v^\nu\rangle_a=\sum_b\int d\mathbf{v}'f_b(v'^\mu)\int d\Omega\sigma(u)u\Delta v^\mu\Delta v^\nu.\quad(4)$$

The differential scattering cross section that we use in Eqs. (3) and (4) is that for an inverse-square law of force,

$$\sigma(\Omega)=(e^4/4m_{ab}^2u^4)[\sin(\theta/2)]^{-4},\quad(5)$$

where $m_{ab}=m_am_b/(m_a+m_b)$ is the reduced mass of the colliding particles and θ is the scattering angle in the center-of-mass system.

[5] Gasiorowicz, Neuman, and Riddell, Phys. Rev. **101**, 922 (1956).

III. DERIVATION OF THE EQUATION

We first discuss the kinematics of the collision between a molecule of type a and belocity $\mathbf{v}$ and a molecule of type b and velocity $\mathbf{v}'$. The relation between $\mathbf{v}_a$, the velocity $\mathbf{V}$ of the center of mass, and the relative velocity $\mathbf{u}=\mathbf{v}-\mathbf{v}'$ is

$$\mathbf{v}_a=\mathbf{V}+\frac{m_b}{m_a+m_b}\mathbf{u}.$$

The change in the μth component of $\mathbf{v}_a$ is given by

$$\Delta v^\mu=\frac{m_b}{m_a+m_b}\Delta u^\mu.\quad(6)$$

We find it convenient to introduce a local Cartesian coordinate system with unit vectors $\mathbf{e}_1'$, $\mathbf{e}_2'$, $\mathbf{e}_3'$ whose relation to the fixed system $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ is given by

$$\mathbf{e}_1'=\frac{\mathbf{u}}{u},\quad\mathbf{e}_2'=\frac{\mathbf{e}_3\times\mathbf{u}}{[(u^1)^2+(u^2)^2]^{\frac{1}{2}}},\quad\mathbf{e}_3'=\mathbf{e}_1'\times\mathbf{e}_2',\quad(7)$$

and in which the relative velocity has components u_L^μ. The changes in the components of u_L^μ produced by a collision are easily calculated in the local Cartesian coordinates, since the relative velocity vector simply undergoes a rotation through an angle θ,

$$\begin{aligned}\Delta u_L^1&=-2u\sin^2(\theta/2),\\ \Delta u_L^2&=2\mu\sin(\theta/2)\cos(\theta/2)\cos\phi,\\ \Delta u_L^3&=2u\sin(\theta/2)\cos(\theta/2)\sin\phi.\end{aligned}\quad(8)$$

A diagram of the scattering is shown in Fig. 1. The changes in the components of the relative velocity $\mathbf{u}$ in the fixed coordinate system are related to these changes in the local system by

$$\begin{aligned}\Delta u^\mu&=(\mathbf{e}_\mu\cdot\mathbf{e}_\nu')\Delta u_L^\nu,\\ \Delta u^\mu\Delta u^\nu&=(\mathbf{e}_\mu\cdot\mathbf{e}_\sigma')(\mathbf{e}_\nu\cdot\mathbf{e}_\omega')\Delta u_L^\sigma\Delta u_L^\omega.\end{aligned}\quad(9)$$

We can next calculate the change of relative velocity in the local system for all collisions by integrating over the scattering angles θ, ϕ, which will be denoted as follows:

$$\{\Delta u_L^\mu\}\equiv\int d\Omega\sigma u(\Delta u_L^\mu).\quad(10)$$

Using Eqs. (5) and (8), we have

$$\{\Delta u_L^1\}=-\left(\frac{\pi e^4}{m_{ab}^2u^2}\right)\int_0^\pi\frac{\sin^2(\theta/2)\sin\theta}{\sin^4(\theta/2)}d\theta,\quad(11)$$

where we have performed the integration over ϕ. The integral diverges logarithmically at small angles, and we therefore introduce a cutoff at $\theta_{\min}$ to obtain

$$\{\Delta u_L^1\} \simeq -(4\pi e^4/m_{ab}^2 u^2)\ln(2/\theta_{\min}). \tag{12}$$

The small-angle deflections correspond to scatterings with very large impact parameters, and the divergence arises from the long-range nature of the Coulomb forces. The divergence is eliminated, however, when we take into account the shielding that arises from the polarization of charge surrounding the scatterer.[6] The polarization screens the scattering particle and provides a natural cutoff on the maximum impact parameter of the order of a Debye length $\lambda_D = (kT/4\pi ne^2)^{\frac{1}{2}}$, and a value for the logarithm in Eq. (12) of

$$\ln(2/\theta_m) = \ln D = \ln(\tfrac{1}{2}m_{ab}u^2)(e^2/\lambda_D)^{-1},$$

$$\lambda_D = (kT/4\pi ne^2 Z_{\text{eff}})^{\frac{1}{2}}.$$

In this equation kT is proportional to the average kinetic energy, n is the number of particles per unit volume, and e is the electronic charge. The quantity D, which is the ratio of the Debye length λ_D to the classical distance of closest approach $(\frac{1}{2}mu^2/e^2)$ for two particles of relative velocity u, in most cases of interest will be a very large number so that $\ln D \gg 1$. From Eqs. (3), (5), and (8) one can easily see that terms of higher order in Δv^μ, like $\langle \Delta v^\mu \Delta v^\nu \Delta v^\omega\rangle$, will not contain $\ln D$, and that the neglect of these terms in the Fokker-Planck equation is therefore justified. The insensitivity of $\ln D$ to the precise value of u means that we can simplify our further development by neglecting the weak dependence on u and using the value for a Maxwell-Boltzmann distribution of $\frac{1}{2}mu^2 \simeq \frac{3}{2}kT$. It would probably not be justified in any event to consider the argument of the logarithm as better determined than this.

The remaining integrations yield

$$\{\Delta u_L^2\}_a = \{\Delta u_L^3\}_a = 0, \tag{13}$$

and

$$\begin{gathered} \{(\Delta u_L^1)^2\} = 0, \\ \{(\Delta u_L^2)^2\} = \{(\Delta u_L^3)^2\} = (4\pi e^4/m_{ab}^2 u^2)\ln D, \end{gathered} \tag{14}$$

with all other second-order terms zero (compared to $\ln D$).

Using these results with Eqs. (6), (7), and (9), we can immediately write down the integrals in the fixed coordinate system,

$$\begin{gathered} \{\Delta v^\mu\}_a = -[4\pi e^4(\ln D)/m_{ab}m_a u^3]u^\mu, \\ \{\Delta v^\mu \Delta v^\nu\}_a = [4\pi e^4(\ln D)/m_a^2]\{\delta^{\mu\nu} - u^\mu u^\nu/(u)^2\}. \end{gathered} \tag{15}$$

These equations can be simplified by noting that

[6] The choice of a cutoff is discussed at length in reference 3.

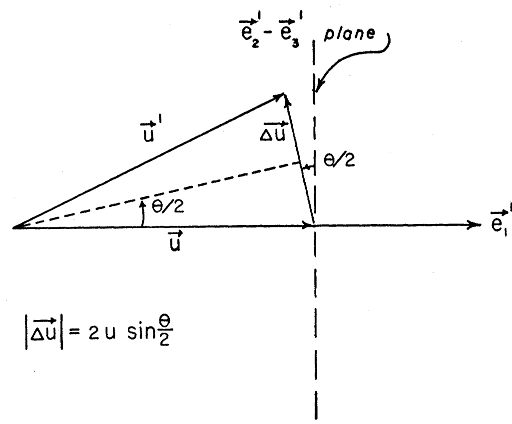

FIG. 1. Diagram showing kinematics of an elastic scatter in the local Cartesian system.

$u = [(v^\mu - v'^\mu)(v^\mu - v'^\mu)]^{\frac{1}{2}}$, so that we have

$$\begin{gathered} \{\Delta v^\mu\} = \Gamma_a \frac{m_a}{m_{ab}}\frac{\partial}{\partial v^\mu}\frac{1}{u}, \quad \{\Delta v^\mu \Delta v^\nu\} = \Gamma_a \frac{\partial^2(u)}{\partial v^\mu \partial v^\nu}, \\ \Gamma_a \equiv (4\pi e^4 \ln D/m_0^2). \end{gathered} \tag{16}$$

Finally, from Eqs. (3), (4), and (16), we obtain

$$\langle \Delta v^\mu\rangle_a = \sum_b \int d\mathbf{v}' f(\mathbf{v}')\{\Delta v^\mu\}_a = \Gamma_a(\partial h_a/\partial v^\mu), \tag{17}$$

$$\langle \Delta v^\mu \Delta v^\nu\rangle_a = \Gamma_a(\partial^2 g/\partial v^\mu \partial v^\nu), \tag{18}$$

where

$$\begin{gathered} h_a(\mathbf{v}) = \sum_b \frac{m_a + m_b}{m_b}\int d\mathbf{v}' f_b(\mathbf{v}')|\mathbf{v} - \mathbf{v}'|^{-1}, \\ g(\mathbf{v}) = \sum_b \int d\mathbf{v}' f_b(\mathbf{v}')|\mathbf{v} - \mathbf{v}'|. \end{gathered} \tag{19}$$

It is interesting to note a formal similarity with potential theory,

$$\begin{gathered} \nabla_v^2 h_a \equiv (\partial^2/\partial v^\mu \partial v^\mu)h_a = -4\pi \sum_b \frac{m_a + m_b}{m_b} f_b(v), \\ \nabla_v^4 g = (\partial^4/\partial v^\mu \partial v^\nu \partial v^\mu \partial v^\nu)g = -8\pi \textstyle\sum_b f_b. \end{gathered} \tag{20}$$

Substituting Eq. (18) into Eq. (2), we obtain the Fokker-Planck equation for an arbitrary distribution function:

$$\frac{1}{\Gamma_a}\left(\frac{\partial f_a}{\partial t}\right)_c = -\frac{\partial}{\partial v^i}\left(f_a \frac{\partial h_a}{\partial v^i}\right) + \frac{1}{2}\frac{\partial^2}{\partial v^i \partial v^j}\left(f_a \frac{\partial^2 g}{\partial v^i \partial v^j}\right). \tag{21}$$

In the general case this fourth-order, three-dimensional, time-dependent, and nonlinear partial differential equation seems quite difficult to handle. In many cases, however, there are simplifications which result when a coordinate system is adopted that embodies the natural symmetries of a problem. For example, in many problems there will be a preferred direction in space, such as the direction of an external applied field, with azimuthal symmetry about this direction. Polar coordinates seem especially suitable for such a problem.

IV. TRANSFORMATION OF THE EQUATION

Although it is possible to transform Eq. (21) by a straightforward change of variables, the procedure is tedious and unnecessarily involved. A much simpler and more direct procedure is to write the equation in a covariant form valid in any set of curvilinear coordinates q^1, q^2, and q^3. Let the expression for distance between two points whose coordinates differ by dq^1, dq^2, and dq^3 be

$$(ds)^2=a_{\mu\nu}dq^\mu dq^\nu,$$

where $a_{\mu\nu}$ is a metric tensor, and let $a^{\mu\nu}\det a$ be the cofactor of $a_{\mu\nu}$ in the matrix $a=(a_{\mu\nu})$, i.e., $a^{\mu\omega}a_{\omega\nu}=\delta_\nu{}^\mu$. We observe that the quantities $T_a{}^\mu=\Gamma_a{}^{-1}\langle\Delta v_a{}^\mu\rangle$ and $S_a{}^{\mu\nu}=\Gamma_a{}^{-1}\langle\Delta v^\mu\Delta v^\nu\rangle_a$ transform like a contravariant vector and tensor, respectively, between different Cartesian coordinate systems. The appropriate tensor extension of Eq. (2) is therefore

$$\Gamma_a{}^{-1}(\partial f_a/\partial t)_c=-(fT_a{}^\mu)_{,\mu}+\tfrac{1}{2}(fS_a{}^{\mu\nu})_{,\mu\nu}, \tag{22}$$

where the commas indicate covariant derivatives with respect to the q^μ. In any Cartesian coordinate system Eq. (22) has precisely the form of Eq. (2). We can now write Eq. (18) in a covariant form,

$$T_a{}^\mu=a^{\mu\nu}(h_a)_{,\nu},\quad S^{\mu\nu}=a^{\mu\omega}a^{\nu\tau}(g_{,\omega\tau}). \tag{23}$$

The two covariant derivatives that appear in Eq. (23) can be found from

$$(h_a)_{,\nu}=\partial h_a/\partial q^\nu,\quad g_{,\omega\tau}=\partial^2g/\partial q^\omega\partial q^\tau-\left\{\begin{matrix}\sigma\\ \omega\tau\end{matrix}\right\}(\partial g/\partial q^\sigma), \tag{24}$$

where $\left\{\begin{matrix}\sigma\\ \omega\tau\end{matrix}\right\}$ is a Christoffel symbol of the second kind defined by

$$\left\{\begin{matrix}\nu\\ \omega\mu\end{matrix}\right\}=a^{\nu\tau}[\omega\mu,\tau]=\tfrac{1}{2}a^{\nu\tau}(\partial a_{\omega\tau}/\partial q^\mu+\partial a_{\mu\tau}/\partial q^\omega-\partial a_{\omega\mu}/\partial q^\tau). \tag{25}$$

The covariant derivative $(fT_a{}^\mu)_{,\mu}$ can be simply expressed

$$(fT_a{}^\mu)_{,\mu}=a^{-\frac{1}{2}}(\partial/\partial q^\mu)(a^{\frac{1}{2}}fT_a{}^\mu),\quad a=\det(a_{\mu\nu}), \tag{26}$$

and for $(fS^{\mu\nu})_{,\mu\nu}$,

$$(fS^{\mu\nu})_{,\mu\nu}=a^{-\frac{1}{2}}(\partial^2/\partial q^\mu\partial q^\nu)(a^{\frac{1}{2}}fS^{\mu\nu})+a^{-\frac{1}{2}}(\partial/\partial q^\nu)\left[a^{\frac{1}{2}}\left\{\begin{matrix}\nu\\ \omega\mu\end{matrix}\right\}fS^{\mu\omega}\right]. \tag{27}$$

The writing of Eq. (22) in arbitrary curvilinear coordinates now becomes a straightforward application of Eqs. (23), (24), (26), and (27), in that order.

As an example we can easily write down the equation in spherical polar coordinates in *velocity space*, assuming azimuthal symmetry about the $\theta=0$ symmetry axis, so that we have $f(v,\mu)$, where $\mu=\cos\theta$. In these coordinates we have

$$\begin{aligned}
&q^1=v,\quad q^2=\mu,\quad q^3=\phi,\\
&ds^2=dv^2+v^2(1-\mu^2)^{-1}(d\mu)^2+v^2(1-\mu^2)(d\phi)^2,\\
&a_{11}=1,\quad a_{22}=v^2(1-\mu^2)^{-1},\quad a_{33}=v^2(1-\mu^2),\\
&\qquad\qquad a_{ij}=0\quad\text{if } i\neq j,\\
&a^{11}=1,\quad a^{22}=v^{-2}(1-\mu^2),\quad a^{33}=v^{-2}(1-\mu^2)^{-1},\\
&\qquad\qquad a^{ij}=0\quad\text{if } i\neq j,\\
&a=\det(a_{\mu\nu})=v^4.
\end{aligned} \tag{28}$$

From Eqs. (23), (24), and (26) we obtain

$$\begin{aligned}
&T_a{}^1=(\partial h_a/\partial v),\quad T_a{}^2=v^{-2}(1-\mu^2)(\partial h/\partial\mu),\quad T_a{}^3=0,\\
&(fT_a{}^\mu)_{,\mu}=v^{-2}(\partial/\partial v)(fv^2\partial h_a/\partial v)\\
&\qquad+v^{-2}(\partial/\partial\mu)[f(1-\mu^2)\partial h_a/\partial\mu].
\end{aligned} \tag{29}$$

The second-rank tensor $S^{\mu\nu}$ follows in the same way:

$$\begin{aligned}
&S^{11}=\partial^2g/\partial v^2,\\
&S^{22}=v^{-4}(1-\mu^2)^2[\partial^2g/\partial\mu^2+v(1-\mu^2)^{-1}(\partial g/\partial v)\\
&\qquad-\mu(1-\mu^2)^{-1}\partial g/\partial\mu],\\
&S^{13}=S^{23}=0,\\
&S^{12}=v^{-2}(1-\mu^2)[\partial^2g/\partial v\partial\mu-v^{-1}\partial g/\partial\mu],\\
&S^{33}=v^{-4}(1-\mu^2)^{-1}[v\partial g/\partial v-\mu\partial g/\partial\mu].
\end{aligned} \tag{30}$$

Using Eq. (26) we calculate the second covariant derivative of $(fS^{\mu\nu})_{,\mu\nu}$ and can then write down Eq. (22) as

$$\begin{aligned}
&\Gamma_a{}^{-1}(\partial f_a/\partial t)_c\\
&\quad=-v^{-2}(\partial/\partial v)(f_av^2\partial h_a/\partial v)-v^{-2}(\partial/\partial\mu)\\
&\quad\times[f_a(1-\mu^2)\partial h_a/\partial\mu]+(2v^2)^{-1}(\partial^2/\partial v^2)\\
&\quad\times(f_av^2\partial^2g/\partial v^2)+(2v^2)^{-1}(\partial^2/\partial\mu^2)\{f_a[v^{-2}(1-\mu^2)^2\\
&\quad\times(\partial^2g/\partial\mu^2)+v^{-1}(1-\mu^2)(\partial g/\partial v)-v^{-2}\mu(1-\mu^2)\\
&\quad\times(\partial g/\partial\mu)]\}+v^{-2}(\partial^2/\partial\mu\partial v)\{f_a(1-\mu^2)\\
&\quad\times[(\partial^2g/\partial\mu\partial v)-v^{-1}(\partial g/\partial\mu)]\}+(2v^2)^{-1}(\partial/\partial v)\\
&\quad\times\{f_a[-v^{-1}(1-\mu^2)(\partial^2g/\partial\mu^2)-2(\partial g/\partial v)\\
&\quad+2\mu v^{-1}(\partial g/\partial\mu)]\}+(2v^2)^{-1}(\partial/\partial\mu)\\
&\quad\times\{f_a[v^{-2}\mu(1-\mu^2)(\partial^2g/\partial\mu^2)+2\mu v^{-1}(\partial g/\partial v)\\
&\qquad+2v^{-1}(1-\mu^2)(\partial^2g/\partial\mu\partial v)-2v^{-2}(\partial g/\partial\mu]\}.
\end{aligned} \tag{31}$$

The equation that describes a system of particles interacting according to an inverse-square law of force when there exists an axis of symmetry is now obtained by combining Eqs. (1), (19), and (31). The quantities h_a and g can be given in terms of two-dimensional integrals,

$$h_a(v,\mu)=\sum_b \frac{m_a+m_b}{m^b}\int_0^\infty dv'v'^2\int_{-1}^1 d\mu' \times f_b(v',\mu')\Lambda(v',\mu';v,\mu), \quad (32)$$

$$g(v,\mu)=\sum_b \int_0^\infty dv'v'^2\int_{-1}^1 d\mu' f_b(v',\mu')\Omega(v',\mu';v,u),$$

with Λ and Ω defined in terms of the complete elliptic integrals K and E as follows:

$$\Lambda=4[v^2+v'^2-2vv'(\mu\mu'-[(1-\mu^2)(1-\mu'^2)]^{\frac{1}{2}})]^{-\frac{1}{2}} \times K\left(\left\{\frac{4vv'[(1-\mu^2)(1-\mu'^2)]^{\frac{1}{2}}}{v^2+v'^2-2vv'[(1-\mu^2)(1-\mu'^2)]^{\frac{1}{2}}}\right\}^{\frac{1}{2}}\right),$$

$$\Omega=4[v^2+v'^2-2vv'(\mu\mu'-[(1-\mu^2)(1-\mu'^2)]^{\frac{1}{2}})]^{\frac{1}{2}} \times E\left(\left\{\frac{4vv'[(1-\mu^2)(1-\mu'^2)]^{\frac{1}{2}}}{v^2+v'^2-2vv'(\mu\mu'-[(1-\mu^2)(1-\mu'^2)]^{\frac{1}{2}}}\right\}^{\frac{1}{2}}\right). \quad (33)$$

The spatially homogeneous two-dimensional time-dependent Eq. (31) is not too complex for electronic digital computers. Moreover, Eq. (31) forms a useful starting point for developing an approximate distribution function when axial symmetry exists. A method for reducing the Eq. (31) to a coupled set of one-dimensional nonlinear integrodifferential equations which can be treated quite simply numerically will be given.

V. REDUCTION OF THE FOKKER-PLANCK EQUATION FOR AXIAL SYMMETRY

The solution to Eq. (31) can be expanded in a series of Legendre polynomials:

$$f_a(v,\mu)=\sum_{n=0}^{\infty} a_n^{(a)}(v)P_n(\mu). \quad (34)$$

This expansion provides an expansion of the two functions $h_a(v,\mu)$ and $g(v,\mu)$, which can be obtained from Eq. (19). We first evaluate the integral appearing in the definition of $h_a(v,\mu)$. Let us define

$$P_n(\mu)A_n^{(a)}(v,\mu)\equiv\int d\mathbf{v}' a_n^{(a)}(v')P_n(\mu')|\mathbf{v}-\mathbf{v}'|^{-1}. \quad (35)$$

Then, inserting

$$|\mathbf{v}-\mathbf{v}'|^{-1}=(2\pi^2)^{-1}\int d\mathbf{k}e^{i\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')}k^{-2}$$

into Eq. (35), we have

$$P_n(\mu)A_n^{(a)}(v,\mu) =(2\pi^2)^{-1}\int d\mathbf{v}'a_n^{(a)}(v')\int d\mathbf{k}e^{i\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')}k^{-2}P_n(\mu'). \quad (36)$$

Writing

$$\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')=\mathbf{k}\cdot\mathbf{v}-kv'[\mu''\mu'+(1-\mu'^2)^{\frac{1}{2}}(1-\mu''^2)^{\frac{1}{2}}\cos\phi],$$

where $\mu''=\cos(\mathbf{k},\mathbf{e}_z)$, $\mu'=\cos(\mathbf{v}',\mathbf{e}_z)$, and ϕ is the angle between the plane of $\mathbf{v}'$ and $\mathbf{e}_z$ and the plane of $\mathbf{k}$ and $\mathbf{e}_z$, we have

$$\int e^{-i\mathbf{k}\cdot\mathbf{v}'}P_n(\mu')d\mu'd\phi'=2\pi\int e^{-ikv'\mu''\mu'} \times J_0(kv'[(1-\mu''^2)(1-\mu'^2)]^1)^{\frac{1}{2}}P_n(\mu')d\mu'. \quad (37)$$

Using a formula given by Watson[7] (12.14) we can finally integrate this to

$$\int e^{-i\mathbf{k}\cdot\mathbf{v}'}P_n(\mu')d\mu'd\phi'=2\pi(2\pi/kv')^{\frac{1}{2}}(-i)^n \times P_n(\mu'')J_{n+\frac{1}{2}}(kv'). \quad (38)$$

If we write $\mathbf{k}\cdot\mathbf{v}=kv\{\mu\mu''+[(1-\mu^2)(1-\mu''^2)]^{\frac{1}{2}}\cos\gamma\}$, where $\mu=\cos(\mathbf{v},\mathbf{e}_z)$ and γ is the angle between the plane of $(\mathbf{k},\mathbf{e}_z)$ and $(\mathbf{v},\mathbf{e}_z)$, we can employ the same formula to integrate Eq. (36) with respect to k, obtaining

$$A_n^{(a)}(v,\mu)=4\pi\int_0^\infty dv'v'^2a_n^{(a)}(v')\int_0^\infty dkJ_{n+\frac{1}{2}}(kv) \times J_{n+\frac{1}{2}}(kv')k^{-1}(vv')^{-\frac{1}{2}}. \quad (39)$$

The integral over k is found in Watson[7] (13.42) also:

$$\int_0^\infty dkJ_{n+\frac{1}{2}}(kv)J_{n+\frac{1}{2}}(kv')k^{-1}=(2n+1)^{-1}(v_</v_>)^{n+\frac{1}{2}},$$

where $v_<$ is the smaller of v, v', and $v_>$ is the greater. Thus the final result is

$$a_n^{(a)}(v,\mu)=4\pi(2n+1)^{-1}\left[\int_0^v dv'\frac{(v')^{n+2}}{v^{n+1}}a_n^{(a)}(v') +\int_v^\infty dv'\frac{v^n}{(v')^{n-1}}a_n^{(a)}(v')\right]. \quad (40)$$

The expansion for $h_a(v,\mu)$ follows from Eqs. (19) and (35):

$$h_a(v,\mu)=\sum_{n=0}\sum_b(m_a+m_b)m_b^{-1}a_n^{(b)}(v,\mu)P_n(\mu). \quad (41)$$

The expansion for $g(v,\mu)$ can be found in the same way by first using

$$|\mathbf{v}-\mathbf{v}'|=-\pi^{-2}\int d\mathbf{k}e^{i\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')}k^{-4}.$$

If we define

$$P_n(\mu)B_n^{(a)}(v)\equiv\int d\mathbf{v}'a_n^{(a)}(\mathbf{v}')\int d\mathbf{k}e^{i\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')}P_n(\mu')k^{-4}, \quad (42)$$

[7] G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, London, 1944), second edition.

the same steps followed above give

$$B_n^{(a)}(v)=2\pi\int_0^\infty dv'(v')^2 a_n^{(a)}(v')\int_0^\infty dkJ_{n+\frac{1}{2}}(kv)\times J_{n+\frac{1}{2}}(kv')k^{-3}(vv')^{-\frac{1}{2}}. \quad (43)$$

The k integral can be evaluated in terms of the hypergeometric function ${}_2F_1(a,b;c;z)$ (reference 7, Sec. 13.4):

$$\int_0^\infty dkJ_{n+\frac{1}{2}}(kv)J_{n+\frac{1}{2}}(kv')k^{-3}=\tfrac{1}{8}(v_<)^{n+\frac{1}{2}}(v_>)^{n-\frac{3}{2}}(n^2-\tfrac{1}{4})^{-1}\times{}_2F_1(n-\tfrac{1}{2},-1;n+\tfrac{3}{2};v_<^2/v_>^2). \quad (44)$$

The hypergeometric function appearing here is actually a polynomial, and the result for $B_n^{(a)}(v)$ is

$$Bn^{(a)}(v)=-4\pi(4n^2-1)^{-1}\times\left[\int_0^v dv'a_n^{(a)}(v')\frac{(v')^{n+2}}{v^{n-1}}\left(1-\frac{n-\frac{1}{2}}{n+\frac{3}{2}}\frac{(v'^2)}{v^2}\right)+\int_v^\infty dv'a_n^{(a)}(v')\frac{v^n}{(v')^{n-3}}\left(1-\frac{n-\frac{1}{2}}{n+\frac{3}{2}}\frac{v^2}{(v')^2}\right)\right]. \quad (45)$$

The expansion for $g(v,\mu)$ follows from Eqs. (19) and (42):

$$g(v,\mu)=\sum_{n=0}^\infty\sum_b B_n^{(b)}(v)P_n(\mu). \quad (46)$$

The procedure for obtaining an approximate solution to Eq. (31) is to retain terms in $f_a(v,\mu)$ to some order N,

$$f_a(v,\mu)\simeq\sum_{n=0}^N a_n^{(a)}(v)P_n(\mu), \quad (47)$$

and obtain the corresponding expansions of $h_a(v,\mu)$ and $g_a(v,\mu)$, which also are to order N. These expressions are now inserted in Eq. (31) and the result expressed as a series in Legendre polynomials. Of use for this purpose is

$$P_{l'}(\mu)P_l(\mu)=\sum_{k=0}^\infty C_{ll'k}P_k(\mu), \quad (48)$$

where the $C_{ll'k}$ are given in Condon and Shortley.[8] Assuming spatial homogeneity, we find that the velocity-dependent term $v_\mu(\partial f_a/\partial v_\mu)$ of the Boltzmann differential operator Eq. (1) can also be expanded in Legendre polynomials. Equating coefficients of Legendre polynomials of the same order in the expansions of Eqs. (1) and (31), one now obtains a system of coupled one-dimensional nonlinear integro-differential equations.

The two simplest approximations are the following (*a*) $f_a(v,\mu)$ in $h_a(v,\mu)$ and $g(v,\mu)$ is isotropic and Eq. (31) is the equation given by Chandrasekhar; (*b*) $f_a(v,\mu)=a_0(v)+a_1(v)P_1(\mu)$, and Eq. (31) is the equation used by Spitzer and collaborators.[3,4]

This work was begun while the first two authors were at the University of California Radiation Laboratory.

[8] E. U. Condon and G. H. Shortley, *Theory of Atomic Spectra* (Cambridge University Press, London, 1935), Sec. 967.