24. THE TRANSPORT EQUATION IN THE CASE OF COULOMB INTERACTIONS

A transport equation is derived for a system consisting of charged particles taking their interactions into account. The order of magnitude of the mean free path of the particles in such a system is determined. The rate at which the temperatures of the ions and electrons in the plasma become equal is evaluated.

In the case of Coulomb interactions there appear, in the formulae for the kinetic theory of gases, integrals which are divergent when the distances between the particles are large. This means that an important role is played by those collisions in which the distances between the colliding particles are large. But at large distances the particles are only scattered through small angles with small changes in velocity. Thus collisions in which the velocity vector is only slightly changed are important.

Let $n(p_i)$ be the distribution function in momentum space. It is a function of the three components of the momentum of the particle ($i = x, y, z$). The change in the momentum during a collision we shall denote by Δ_i where $\Delta_i \ll p_i$ in all the collisions. Further, let $\mathrm{d}W$ be the probability (per unit time) of a collision between particles with momentum p_i and a particle with momentum p_i', such that p_i' is changed to $p_i + \Delta_i$ and p_i' to $p_i' + \Delta_i'$. Because of momentum conservation $\Delta_i = -\Delta_i'$. We shall not, however, use this fact for the moment, in order that we may obtain formulae which are valid in the general case. The number of such collisions will then be

$$\mathrm{d}W\, n(p)\, n'(p')$$

(for simplicity we shall omit the indices on p_i and Δ_i in $n(p_i)$ and so on).

The number of collisions changing particle momenta $p_i + \Delta_i$ and $p_i' + \Delta_i'$ back to p_i and p_i' will equal

$$\mathrm{d}W\, n(p+\Delta)\, n(p'+\Delta'),$$

since according to the Liouville theorem the probabilities of forward and reverse transitions are equal.

Let us express the probability $\mathrm{d}W$ as a function of the half-sum and half-difference of the momenta in the initial and final states. Then the probability of a forward transition will be

$$\mathrm{d}W\left(p + \frac{\Delta}{2},\ p' + \frac{\Delta'}{2},\ \Delta,\ \Delta'\right),$$

Л. Д. Ландау, Кинетическое уравнение в случае кулоновского взаимодействия, *Журнал Экспериментальной и Теоретической Физики*, **7**, 203 (1937).

L. Landau, Die kinetische Gleichung für den Fall Coulombscher Wechselwirkung, *Phys. Z. Sowjet.* **10**, 154 (1936).

and for the reverse transition

$$\mathrm{d}\,W\left(p + \frac{\Delta}{2},\ p' + \frac{\Delta'}{2},\ -\Delta,\ -\Delta'\right).$$

Since these probabilities are equal, $\mathrm{d}\,W(p, p^1, \Delta, \Delta^1)$ is an even function of Δ_i and Δ'_i.

Hence the number of particles with momentum p_i is changed, due to collisions, in unit time by

$$\int \mathrm{d}\,W\left(p + \frac{\Delta}{2},\ p' + \frac{\Delta'}{2},\ \Delta,\ \Delta'\right)\{n(p)\,n'(p') - n(p+\Delta)\,n'(p'+\Delta')\}.$$

The probability $\mathrm{d}\,W$ we write in the form

$$\mathrm{d}\,W = w\left(p + \frac{\Delta}{2},\ p' + \frac{\Delta'}{2},\ \Delta,\ \Delta'\right)\mathrm{d}\,\tau'\,\mathrm{d}\,\tau_\Delta,$$

where $\mathrm{d}\,\tau' = \mathrm{d}p'_x\,\mathrm{d}p'_y\,\mathrm{d}p'_z$ and $\mathrm{d}\,\tau_\Delta$ is the product of the differentials of the parameters which define the collision.

Thus the change in the number of particles with momentum p'_i is:

$$\int \mathrm{d}\,\tau'\,\mathrm{d}\,\tau_\Delta\,w\left(p + \frac{\Delta}{2},\ p' + \frac{\Delta'}{2},\ \Delta,\ \Delta'\right)\{n(p)\,n'(p') - n(p+\Delta)\,n'(p'+\Delta')\}. \tag{1}$$

Let us expand the expression under the integral in a series in powers of Δ_i and Δ'_i (ω should of course be expanded only with respect to Δ_i, appearing in $p_i + \Delta_i/2$ and $p'_i + \Delta'_i/2$). The zero order terms cancel each other and the terms of the first order are

$$-\int \mathrm{d}\,\tau'\,\mathrm{d}\,\tau_\Delta\left(w\,n'\frac{\partial n}{\partial p_i}\Delta_i + w\,n\frac{\partial n'}{\partial p'_i}\Delta'_i\right),$$

where $w = w(p, p', \Delta, \Delta')$ (summation is everywhere implied over indices which are repeated twice). But w is an even function of Δ_i and Δ'_i. Therefore the integral written above is equal to zero.

The second order terms are the following

$$\begin{aligned}
&-\int \mathrm{d}\,\tau'\,\mathrm{d}\,\tau_\Delta\,w\left\{\frac{\Delta_i\,\Delta_k}{2}n'\frac{\partial^2 n}{\partial p_i\,\partial p_k} + \Delta_i\,\Delta'_k\frac{\partial n'}{\partial p'_k}\frac{\partial n}{\partial p_i} + \frac{\Delta'_i\,\Delta'_k}{2}n\frac{\partial^2 n'}{\partial p'_i\,\partial p'_k}\right\}\\
&-\int \mathrm{d}\,\tau'\,\mathrm{d}\,\tau_\Delta\,w\frac{1}{2}\left(\Delta_i\frac{\partial w}{\partial p_i} + \Delta'_i\frac{\partial w}{\partial p'_i}\right)\left(\Delta_k\,n'\frac{\partial n}{\partial p_k} + n\,\Delta'_k\frac{\partial n'}{\partial p'_k}\right).
\end{aligned} \tag{2}$$

Let us integrate two of these terms by parts over $\mathrm{d}\,\tau'$, namely:

$$-\frac{1}{2}\int \mathrm{d}\,\tau'\,\mathrm{d}\,\tau_\Delta\,\Delta'_i\,\Delta_k\,n'\frac{\partial w}{\partial p'_i}\frac{\partial n}{\partial p'_k} = \frac{1}{2}\int \mathrm{d}\,\tau'\,\mathrm{d}\,\tau_\Delta\,\Delta'_i\,\Delta_k\,w\frac{\partial n'}{\partial p'_i}\frac{\partial n}{\partial p_k}$$

$$-\frac{1}{2}\int \mathrm{d}\,\tau'\,\mathrm{d}\,\tau_\Delta\,\Delta'_i\,\Delta'_k\frac{\partial w}{\partial p'_i}\frac{\partial n'}{\partial p'_k}n = \frac{1}{2}\int \mathrm{d}\,\tau'\,\mathrm{d}\,\tau_\Delta\,\Delta'_i\,\Delta'_k\,w\frac{\partial^2 n'}{\partial p'_i\,\partial p'_k}n$$

Since the integration is performed over the whole of p' space, the surface integral is equal to zero, because $n' = 0$ at infinity.

As a result the second-order terms give

$$-\int \mathrm{d}\tau'\,\mathrm{d}\tau_\Delta\, w\left\{\frac{\Delta_i\,\Delta_k}{2}\,n'\,\frac{\partial^2 n}{\partial p_i\,\partial p_k}+\frac{\Delta_i\,\Delta'_k}{2}\,\frac{\partial n}{\partial p_i}\,\frac{\partial n'}{\partial p'_k}\right\}$$
$$-\int \mathrm{d}\tau'\,\mathrm{d}\tau_\Delta\, w\left\{\frac{\Delta_i\,\Delta_k}{2}\,\frac{\partial w}{\partial p_i}\,\frac{\partial n}{\partial p_k}\,n'+\frac{\Delta_i\,\Delta'_k}{2}\,n\,\frac{\partial w}{\partial p_i}\,\frac{\partial n'}{\partial p'_k}\right\}.$$

This can be re-written in the form

$$-\frac{\partial}{\partial p_i}\int \mathrm{d}\tau'\,\mathrm{d}\tau_\Delta\, w\left\{\frac{\Delta_i\,\Delta_k}{2}\,n'\,\frac{\partial n}{\partial p_k}+\frac{\Delta_i\,\Delta'_k}{2}\,n\,\frac{\partial n'}{\partial p'_k}\right\}.$$

Thus the integral (1), defining the change due to collisions in the number of particles with given momentum is expressed, as it should be, as the divergence $\partial j_i/\partial p_i$ in momentum space, of the flow vector j_i in momentum space. The components of this flow equal

$$j_i = -\int \mathrm{d}\tau'\,\mathrm{d}\tau_\Delta\, w\left\{\frac{\Delta_i\,\Delta_k}{2}\,n'\,\frac{\partial n}{\partial p_k}+\frac{\Delta_i\,\Delta'_k}{2}\,n\,\frac{\partial n'}{\partial p'_k}\right\}.$$

As was already noted at the beginning, $\Delta_i = -\Delta'_i$. Therefore in our case the flow is

$$j_i = \int \mathrm{d}\tau'\,\mathrm{d}\tau_\Delta\left\{\left(n\,\frac{\partial n'}{\partial p'_k}-n'\,\frac{\partial n}{\partial p_k}\right)\int\frac{\Delta_i\,\Delta_k}{2}\,w\,\mathrm{d}\tau_\Delta\right\}.$$

If the system consists of different types of particles, then the flow j_i for a given type of particle is equal to

$$j_i = \sum\int \mathrm{d}\tau'\left\{\left(n\,\frac{\partial n'}{\partial p'_k}-n'\,\frac{\partial n}{\partial p_k}\right)\int\frac{\Delta_i\,\Delta_k}{2}\,w\,\mathrm{d}\tau_\Delta\right\}, \tag{3}$$

where the summation is performed over all the kinds of particles in the system, unprimed variables being related to the given type of particle and primed variables to each type of particle in turn (in this number, of course, is included the given type).

Let us apply the formulae thus obtained to the case of a system of particles with Coulomb interactions, which we are considering. For this system let us determine the change in the momenta of two particles with charges, e and e' and momenta p_i and p'_i moving at some distance from one another. Let ϱ be the impact parameter, i.e. the distance at which the two particles would pass each other if there were no interaction between them, and u_i their relative velocity. Let us consider this collision in the co-ordinate system in which the particle e' is at rest, with the x-axis along the direction of motion of the particle e, which has velocity u. We consider the scattering angle to be small. Because of this the momentum along the x-axis does not change to this approximation, and only the momentum in a direction perpendicular to the x-axis (along the y-axis) changes. This change equals

$$\Delta_y = \int_{-\infty}^{+\infty} -\frac{\partial U}{\partial y} \mathrm{d}t,$$

where $U = e\,e'/r$ is the energy of interaction between the particles.

Since the scattering is considered to be small it is possible to consider, in the integral, that the motion is unperturbed, i.e. directed along the x-axis. Then

$$\Delta_y = \int_{-\infty}^{+\infty} \frac{e\,e'\,\varrho\,\mathrm{d}t}{(\varrho^2 + u^2 t^2)^{3/2}} = \frac{2e\,e'}{\varrho\,u}.$$

Going back to an arbitrary co-ordinate system, and noticing that the vector of the change in momentum is directed along the direction of ϱ_i we find

$$\Delta_i = \frac{2e\,e'}{u} \frac{\varrho_i}{\varrho^2}. \tag{4}$$

Let us now calculate the integrals

$$\alpha_{ik} = \int \frac{\Delta_i \Delta_k}{2} w\,\mathrm{d}\tau_\Delta = \int \frac{2e^2 e'^2}{u^2} \frac{\varrho_i \varrho_k}{\varrho^4} w\,\mathrm{d}\tau_\Delta,$$

appearing in (3). $n'\,\mathrm{d}W = \omega\, n'\,\mathrm{d}\tau'\,\mathrm{d}\tau_\Delta$ is the number of collisions per unit time with particles e', undergone by the particle e with momentum p_i, in which its momentum changes by the given value Δ_i. In other words this is the number of collisions in which particles e and e' pass a definite distance ϱ_i apart, the particles e' having definite momentum p'_i (Δ_i is completely determined for given p'_i and ϱ_i). Denote by v_i and v'_i the velocities of the particles e and e'. Their relative velocity $u_i = v_i - v'_i$ has absolute value u. The number of collisions of the particle e which take place at a given distance ϱ_i with the given relative velocity u_i is obviously

$$u\,\varrho\,\mathrm{d}\varrho\,\mathrm{d}\varphi\,u'\,\mathrm{d}\tau',$$

where φ is the angle determining the direction of ϱ_i (at the given velocity $\boldsymbol{u}_i$ all the possible $\boldsymbol{\varrho}_i$ lie in one plane which is perpendicular to $\boldsymbol{u}_i$; φ is the angle in that plane).

Hence we can change $w\,\mathrm{d}\tau_\Delta$ to $u\,\varrho\,\mathrm{d}\varrho\,\mathrm{d}\varphi$ in the integrals α_{ik}

$$\alpha_{ik} = \frac{2e^2 e'^2}{u} \int \frac{\varrho_i \varrho_k}{\varrho^3} \mathrm{d}\varrho\,\mathrm{d}\varphi.$$

In order to perform the integration, introduce, temporarily, co-ordinate axes with the x-axis directed along $\boldsymbol{u}_i$. Then $\varrho_x = 0$ since $\boldsymbol{\varrho}_i \perp \boldsymbol{u}_i$. Because of this $\alpha_{xx} = \alpha_{xy} = \alpha_{xz} = 0$. Also $\alpha_{yz} = 0$ since the integral of $\varrho_y \varrho_z = \varrho^2 \sin\varphi \cos\varphi$ over all angles φ vanishes.

Thus for α_{yy} and α_{zz}, which are not equal to zero, we find (substituting $\varrho_z = \varrho \sin\varphi$, $\varrho_y = \varrho \cos\varphi$ and integrating with respect to $\mathrm{d}\varphi$)

$$\alpha_{yy} = \alpha_{zz} = \frac{2\pi e^2 e'^2}{u} \int_{\varrho_1}^{\varrho_2} \frac{\mathrm{d}\varrho}{\varrho}. \tag{5}$$

The integral appearing here diverges logarithmically. The divergence at small ϱ is due to the fact that for small ϱ the scattering angle of the particles in the collision is large, and hence all the previous formulae are no longer valid. If the exact formulae are used then there would, of course, be no divergence (at small ϱ).

Since a logarithm is insensitive to small changes in its argument, we can take in (5), as the lower limit ϱ_1, that value ϱ at which the scattering angle becomes of the order of unity, i.e. the interaction energy $e\,e'/\varrho$ becomes of the order of the mean kinetic energy $\bar{\varepsilon}$ of the particles:

$$\varrho_1 = \frac{e\,e'}{\bar{\varepsilon}}.$$

As far as the upper limit ϱ_2 in (5) is concerned, two cases must be distinguished. If the total charge on the particles in the system is not equal to zero, then as the upper limit one must take the linear dimension R of the region in which these particles lie. In the most interesting case, when the total charge of the system is zero, the charges are screened and as ϱ_2 one should take the Debye–Hückel screening radius. This radius is $1/\varkappa$ where $\varkappa$ is the coefficient in the screened Coulomb law $e^{-\varkappa r}/r$ and is determined by the well-known equation

$$\varkappa^2 = \sum \frac{N_i\, e_i^2}{k\,T}.$$

Here the summation is taken over all types of particles in the system and N_i is the number of particles of the ith kind in 1 cm³. To an order of magnitude $\varkappa \cong \sqrt{\bar{N}\, e^2/kT}$ where $\bar{N}$ is the number of particles in 1 cm³. But $kT \cong \bar{\varepsilon}$ so that $\varkappa = \sqrt{\bar{N}\, e^2/\bar{\varepsilon}}$. Thus we can take for the upper limit in (5),

$$\varrho_2 = \sqrt{\frac{\bar{\varepsilon}}{\bar{N}\, e^2}}.$$

Substituting ϱ_1 and ϱ_2 in (5) we find

$$\alpha_{yy} = \alpha_{zz} = \frac{\pi\, e^2\, e'^2}{u} L,$$

where

$$L = \ln \frac{1}{\bar{N}} \left(\frac{\bar{\varepsilon}}{e^2}\right)^3. \tag{6}$$

Returning now to an arbitrary co-ordinate system we can write, in tensor form,

$$\alpha_{ik} = \pi e^2 e'^2 L \frac{u^2 \delta_{ik} - u_i u_k}{u^3},$$

where

$$\delta_{ik} = \begin{cases} 1, & i = k, \\ 0, & i \neq k. \end{cases}$$

Substituting this expression into (3) we find the flow of particles e in momentum space in the form

$$j_i = \pi e^2 L \sum e'^2 \int \left\{ n \frac{\partial n'}{\partial p'_k} - n' \frac{\partial n}{\partial p_k} \right\} \frac{u^2 \delta_{ik} - u_i u_k}{u^3} \, \mathrm{d}\tau'. \tag{7}$$

The transport equation in the presence of a temperature gradient and an external electric field E_i has the form

$$\frac{\partial n}{\partial t} + \frac{\partial n}{\partial T} v_i \frac{\partial T}{\partial x_i} + e E_i \frac{\partial n}{\partial p_i} + \frac{\partial j_i}{\partial v_i} = 0. \tag{8}$$

The Maxwellian distribution makes j_i zero, as it should do.

It would, in principle, be possible to determine from this equation the electrical and thermal conductivity of the gas consisting of the charged particles. This, however, meets considerable mathematical difficulties. We restrict ourselves to a qualitative determination of the conductivities, namely, we determine, to within an order of magnitude, the mean free path l of the particles, from which it is possible to find the electrical and thermal conductivities by the use of well-known formulae.

Let $\overline{N}$ be (to an order of magnitude) the number of particles in 1 cm^3, e the charge of the particles and T the temperature of the gas. As is seen from (7), when it is substituted into (8), $\overline{N}$ and e appear in the formulae only in the combination $\overline{N} L e^4$. Therefore, the mean free path of the particles should be determined only in terms of the quantities $e^4 L \overline{N}$, kT and the mass of the particles. From these one can construct only one combination having the dimensions of a length, namely $(kT)^2/(e^4 L \overline{N})$. To within an order of magnitude the mean free path will be equal to just this ratio

$$l \cong \frac{k^2 T^2}{e^4 L \overline{N}}. \tag{9}$$

This result disagrees with Gabor's formulae[1], which points to the incorrectness of his assumptions.

Let us consider a gas consisting of electrons and ions. Because of the large difference in masses between the electrons and ions, the exchange of energy by the electrons amongst themselves and the ions amongst themselves will take place much more rapidly than the exchange of energy between the electrons and ions (in a collision between a very heavy particle and a very light one, the energy of each of them is almost unchanged). Because of this the equilibrium in the energies of the electrons amongst themselves and the ions amongst themselves will be established much sooner than the equilibrium between the unlike groups. Let us consider that such an equilibrium is already established, i.e. the electrons and the ions both have a Maxwellian distribution, but the temperatures of these distributions, T' and T, are different. Let us find the rate at which the equilibrium between the electrons and ions is established, i.e. the rate of equalisation of the temperatures T' and T.

Let us work out the energy transmitted by the electrons to the ions in unit time (in 1 cm³) by collisions between them. Let e, m and e', m' be the charges and masses of the ions and electrons and n and n' their distributions:

$$n = N(2\pi\, m\, k\, T)^{-3/2}\, \mathrm{e}^{-\varepsilon/kT}, \quad n' = N'\,(2\pi\, m'\, k\, T)^{-3/2}\, \mathrm{e}^{-\varepsilon'/kT}; \tag{10}$$

N and N' are the numbers of ions and electrons in 1 cm³ and ε and ε' are their energies. The flow of ions in momentum space is, according to (7):

$$j_i = \pi\, e^2\, e'^2\, L \int \left(n \frac{\partial n'}{\partial p'_k} - n' \frac{\partial n}{\partial p_k} \right) \frac{u^2\, \delta_{ik} - u_i\, u_k}{u^3}\, \mathrm{d}\tau' \tag{11}$$

(all primed variables correspond to the electrons, unprimed variables to the ions). In the sum in (7) only one term remains, since the term which corresponds to the collisions of ions one with another vanishes, because the distribution of the ions is Maxwellian.

The change per unit time in the number of ions with given momenta due to collisions with electrons is $-\partial j_i/\partial p_i$. Thus the change in their energy is

$$-\int \varepsilon \frac{\partial j_i}{\partial p_i}\, \mathrm{d}\tau,$$

or, integrating by parts

$$-\int \varepsilon \frac{\partial j_i}{\partial p_i}\, \mathrm{d}\tau = \int j_i \frac{\partial \varepsilon}{\partial p_i}\, \mathrm{d}\tau = \int j_i\, v_i\, \mathrm{d}\tau$$

($\partial \varepsilon/\partial p_i = v_i$). Since the integration is taken over all momentum space, the surface integral disappears.

Substitute the distributions (10) into (11). We have

$$\frac{\partial n}{\partial p_k} = -\frac{n}{kT} \frac{\partial \varepsilon}{\partial p_k} = -\frac{n\, v_k}{kT}, \quad \frac{\partial n'}{\partial p'_k} = -\frac{n'\, v'_k}{kT}.$$

Then we find

$$j_i = \pi\, e^2\, e'^2\, L \int n\, n' \left(\frac{v_k}{kT} - \frac{v'_k}{kT'} \right) \frac{u^2\, \delta_{ik} - u_i\, u_k}{u^3}\, \mathrm{d}\tau'$$

$$= \pi\, e^2\, e'^2\, L \int n\, n' \left[v_k \left(\frac{1}{kT} - \frac{1}{kT'} \right) + \frac{u_k}{kT'} \right] \frac{u^2\, \delta_{ik} - u_i\, u_k}{u^3}\, \mathrm{d}\tau'.$$

But

$$u_k \frac{u^2\, \delta_{ik} - u_i\, u_k}{u^3} = 0$$

and therefore

$$j_i = \pi e^2 e'^2 L\left(\frac{1}{kT} - \frac{1}{kT'}\right)\int n\,n'\,\frac{u^2 v_i - (u_i v_i)^2}{u^3}\,\mathrm{d}\tau\,\mathrm{d}\tau'.$$

The change in energy, which we are seeking, is then equal to

$$\int j_i v_i \,\mathrm{d}\tau = \pi e^2 e'^2 L\left(\frac{1}{kT} - \frac{1}{kT'}\right)\int\int n\,n'\,\frac{u_i^2 v_i^2 - (v_i u_i)^2}{u^3}\,\mathrm{d}\tau\,\mathrm{d}\tau'.$$

Since the mass of the electrons is much less than the mass of the nuclei, their velocity v'_k is much larger than the velocity of the ions v_k. Because of this one may consider that $u_i \cong v'_i$. Then

$$\int j_i v_i \,\mathrm{d}\tau = \pi e^2 e'^2 L\left(\frac{1}{kT} - \frac{1}{kT'}\right)\int\int n\,n'\,\frac{v_i^2 v_i'^2 - (v_i v'_i)^2}{v'^3}\,\mathrm{d}\tau\,\mathrm{d}\tau'.$$

Averaging over the angles between v_i and v'_i we find

$$\int j_i v_i \,\mathrm{d}\tau = \frac{2}{3}\pi e^2 e'^2 L\left(\frac{1}{kT} - \frac{1}{kT'}\right)\int n\,v^2\,\mathrm{d}\tau \int \frac{n'}{v'}\,\mathrm{d}\tau'.$$

Substituting (10) we have:

$$\int n\,v^2\,\mathrm{d}\tau = N\frac{3kT}{m}, \quad \int \frac{n'}{v'}\,\mathrm{d}\tau' = 4\pi N'\left(\frac{m'}{2\pi kT'}\right)^{3/2}\int_0^\infty e^{-\frac{m'v'^2}{2kT'}}\,v'\,\mathrm{d}v'$$
$$= 2N'\sqrt{\frac{m'}{2\pi kT'}}.$$

As a result we find:

$$\int v_i j_i \,\mathrm{d}\tau = \frac{2NN' e^2 e'^2 (2\pi m')^{1/2} L}{m\,k^{1/2}\,T'^{3/2}}(T' - T).$$

If there are ions of different types in the gas, the total energy transmitted by the electrons to the ions per unit time is

$$\frac{2N' e'^2 (2\pi m')^{1/2} L}{k^{1/2}\,T'^{3/2}}(T' - T)\sum\frac{N e^2}{m} \tag{12}$$

($\sum$ is over all types of ions).

The energy of the electrons in 1 cm^3 is equal to $3N'kT'/2$. Dividing the energy (12), lost by the electrons in unit time, by $3N'k/2$, we obtain the rate of change of the electron temperature T':

$$\frac{\mathrm{d}T'}{\mathrm{d}t} = -\frac{4}{3}\,\frac{e'^2(2\pi m')^{1/2}(T' - T)}{(kT')^{3/2}}\,L\sum\frac{N e^2}{m}. \tag{13}$$

Reference

1. D. Gabor, *Phys. Z.* **34**, 38 (1933).