THE EIGENSYSTEM OF THE TEN-MOMENT EQUATIONS

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1. The eigensystem of the equations written in quasilinear form

In this document I list the eigensystem of the ten-moment equations. These equations are derived by taking moments of the Boltzmann equation and truncating the resulting infinite series of equations by assuming the heat flux tensor vanishes. In non-conservative form these equations are

$$\partial_t n + n\partial_j u_j + u_j \partial_j n = 0 \tag{1}$$

$$\partial_t u_i + \frac{1}{mn}\partial_j P_{ij} + u_j \partial_j u_i = \frac{q}{m}\left(E_i + \epsilon_{kmi} u_k B_m\right) \tag{2}$$

$$\partial_t P_{ij} + P_{ij}\partial_k u_k + \partial_k u_{[i} P_{j]k} + u_k \partial_k P_{ij} = \frac{q}{m} B_m \epsilon_{km[i} P_{jk]} \tag{3}$$

In these equations square brackets around indices represent the minimal sum over permutations of free indices within the bracket needed to yield completely symmetric tensors. Note that there is one such system of equations for *each* species in the plasma. Here, q is the species charge, m is the species mass, n is the number density, u_j is the velocity, P_{ij} the pressure tensor and $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic field respectively. Also $\partial_t \equiv \partial/\partial t$ and $\partial_i \equiv \partial/\partial x_i$.

To determine the eigensystem of the homogeneous part of this system we first write, in one-dimension, the left-hand side of Eqns. (1)-(3) in the form

$$\partial_t \mathbf{v} + \mathbf{A}\partial_1 \mathbf{v} = 0 \tag{4}$$

where $\mathbf{v}$ is the vector of primitive variables and $\mathbf{A}$ is the quasilinear coefficient matrix[1]. For the ten-moment system we have

$$\mathbf{v} = \left[\rho, u_1, u_2, u_3, P_{11}, P_{12}, P_{13}, P_{22}, P_{23}, P_{33}\right]^T \tag{5}$$

[1]There is no standard name for this matrix. I choose to call it the *quasilinear coefficient matrix* instead of the incorrect term "primitive flux Jacobian".

where $\rho \equiv mn$ and

$$
\mathbf{A} = \begin{bmatrix}
u_1 & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & u_1 & 0 & 0 & 1/\rho & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & u_1 & 0 & 0 & 1/\rho & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & u_1 & 0 & 0 & 1/\rho & 0 & 0 & 0 \\
0 & 3P_{11} & 0 & 0 & u_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2P_{12} & P_{11} & 0 & 0 & u_1 & 0 & 0 & 0 & 0 \\
0 & 2P_{13} & 0 & P_{11} & 0 & 0 & u_1 & 0 & 0 & 0 \\
0 & P_{22} & 2P_{12} & 0 & 0 & 0 & 0 & u_1 & 0 & 0 \\
0 & P_{23} & P_{13} & P_{12} & 0 & 0 & 0 & 0 & u_1 & 0 \\
0 & P_{33} & 0 & 2P_{13} & 0 & 0 & 0 & 0 & 0 & u_1
\end{bmatrix} \tag{6}
$$

The eigensystem of this matrix needs to be determined. It is easiest to use a computer algebra system for this. I prefer the open source package Maxima for this. The right-eigenvectors returned by Maxima need to massaged a little bit to bring them into a clean form. The results are described below.

The eigenvalues of the system are given by

$$\lambda^{1,2} = u_1 - \sqrt{P_{11}/\rho} \tag{7}$$

$$\lambda^{3,4} = u_1 + \sqrt{P_{11}/\rho} \tag{8}$$

$$\lambda^{5} = u_1 - \sqrt{3P_{11}/\rho} \tag{9}$$

$$\lambda^{6} = u_1 + \sqrt{3P_{11}/\rho} \tag{10}$$

$$\lambda^{7,8,9,10} = u_1 \tag{11}$$

To maintain hyperbolicity we must hence have $\rho > 0$ and $P_{11} > 0$. In multiple dimensions, in general, the diagonal elements of the pressure tensor must be positive. When $P_{11} = 0$ the system reduces to the cold fluid equations which is known to be rank deficient and hence not hyperbolic as usually understood[2]. Also notice that the eigenvalues do not include the usual fluid sound-speed $c_s = \sqrt{5p/3\rho}$ but instead have two different propagation speeds $c_1 = \sqrt{P_{11}/\rho}$ and $c_2 = \sqrt{3P_{11}/\rho}$. This is because the (neutral) ten-moment system does not go to the correct limit of Euler equations in the absence of collisions. In fact, it is collisions that drive the pressure tensor to isotropy. These collision terms should also be included in the plasma ten-moment system. In this case, however, the situation is complicated due to the presence of multiple species of very different masses which leads to inter-species collision terms that need to be computed carefully. For a two-species plasma, for example, see the paper by Green[1] in which the relations for relaxation of momentum and energy are used to derive a simplified collision integral for use in the Boltzmann equation.

[2]For hyperbolicity the quasilinear matrix must posses real eigenvalues and a complete set of linearly independent right eigenvectors. For the cold fluid system we only have a single eigenvalue (the fluid velocity) and a single eigenvector. This can lead to generalized solutions like delta shocks.

The right eigenvectors (column vectors) are given below.

$$\mathbf{r}^{1,3} = \begin{bmatrix} 0 \\ 0 \\ \mp c_1 \\ 0 \\ 0 \\ P_{11} \\ 0 \\ 2P_{12} \\ P_{13} \\ 0 \end{bmatrix} \quad \mathbf{r}^{2,4} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \mp c_1 \\ 0 \\ 0 \\ P_{11} \\ 0 \\ P_{12} \\ 2P_{13} \end{bmatrix} \tag{12}$$

and

$$\mathbf{r}^{5,6} = \begin{bmatrix} \rho P_{11} \\ \mp c_2 P_{11} \\ \mp c_2 P_{12} \\ \mp c_2 P_{13} \\ 3P_{11}^2 \\ 3P_{11}P_{12} \\ 3P_{11}P_{13} \\ P_{11}P_{22} + 2P_{12}^2 \\ P_{11}P_{23} + 2P_{12}P_{13} \\ P_{11}P_{33} + 2P_{13}^2 \end{bmatrix} \tag{13}$$

and

$$\mathbf{r}^{7} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{r}^{8} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{r}^{9} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{r}^{10} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{14}$$

We can now compute the left eigenvectors (row vectors) by inverting the matrix with right eigenvectors stored as columns. This ensures the normalization $\mathbf{l}^p\mathbf{r}^k = \delta^{pk}$, where the $\mathbf{l}^p$ are the left eigenvectors. On performing the inversion we have

$$\mathbf{l}^{1,3} = \begin{bmatrix} 0 & \pm\dfrac{P_{12}}{2c_1P_{11}} & \mp\dfrac{1}{2c_1} & 0 & -\dfrac{P_{12}}{2P_{11}^2} & \dfrac{1}{2P_{11}} & 0 & 0 & 0 & 0 \end{bmatrix} \tag{15}$$

$$\mathbf{l}^{2,4} = \begin{bmatrix} 0 & \pm\dfrac{P_{13}}{2c_1P_{11}} & 0 & \mp\dfrac{1}{2c_1} & -\dfrac{P_{13}}{2P_{11}^2} & 0 & \dfrac{1}{2P_{11}} & 0 & 0 & 0 \end{bmatrix} \tag{16}$$

and

$$\mathbf{l}^{5,6} = \begin{bmatrix} 0 & \mp\dfrac{1}{2c_2P_{11}} & 0 & 0 & \dfrac{1}{6P_{11}^2} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{17}$$

and

$$\mathbf{l}^{7} = \begin{bmatrix} 1 & 0 & 0 & 0 & -\dfrac{1}{3c_1^2} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{18}$$

$$\mathbf{l}^{8} = \begin{bmatrix} 0 & 0 & 0 & 0 & \dfrac{4P_{12}^2 - P_{11}P_{22}}{3P_{11}^2} & -\dfrac{2P_{12}}{P_{11}} & 0 & 1 & 0 & 0 \end{bmatrix} \tag{19}$$

$$\mathbf{l}^{9} = \begin{bmatrix} 0 & 0 & 0 & 0 & \dfrac{4P_{12}P_{13} - P_{11}P_{23}}{3P_{11}^2} & -\dfrac{P_{13}}{P_{11}} & -\dfrac{P_{12}}{P_{11}} & 0 & 1 & 0 \end{bmatrix} \tag{20}$$

$$\mathbf{l}^{10} = \begin{bmatrix} 0 & 0 & 0 & 0 & \dfrac{4P_{13}^2 - P_{11}P_{33}}{3P_{11}^2} & 0 & -\dfrac{2P_{13}}{P_{11}} & 0 & 0 & 1 \end{bmatrix} \tag{21}$$

2. The eigensystem of the equations written in conservative form

In the wave-propagation scheme the quasilinear equations can be updated. However, the resulting solution will not be conservative. This actually might not be a problem for the ten-moment system as the system (as written) can not be put into a homogeneous conservation law form anyway. However, most often for numerical simulations the eigensystem of the conservation form of the homogeneous system is needed. This eigensystem is related to the eigensystem of the quasilinear form derived above. To see this consider a conservation law

$$\partial_t \mathbf{q} + \partial_1 \mathbf{f} = 0 \tag{22}$$

where $\mathbf{f} = \mathbf{f}(\mathbf{q})$ is a flux function. Now consider an invertible transformation $\mathbf{q} = \varphi(\mathbf{v})$. This transforms the conservation law to

$$\partial_t \mathbf{v} + (\varphi')^{-1}\ D\mathbf{f}\ \varphi' \partial_1 \mathbf{v} = 0 \tag{23}$$

where φ' is the Jacobian matrix of the transformation and $D\mathbf{f} \equiv \partial \mathbf{f}/\partial \mathbf{q}$ is the flux Jacobian. Comparing this to Eq. (4) we see that the quasilinear matrix is related to the flux Jacobian by

$$\mathbf{A} = (\varphi')^{-1}\ D\mathbf{f}\ \varphi' \tag{24}$$

This clearly shows that the eigenvalues of the flux Jacobian are the same as those of the quasilinear matrix while the right and left eigenvectors can be computed using $\varphi' \mathbf{r}^p$ and $\mathbf{l}^p (\varphi')^{-1}$ respectively.

For the ten-moment system as written in Eqns. (1)-(3) the required transformation is

$$\mathbf{q} = \varphi(\mathbf{v}) = \begin{bmatrix} \rho \\ \rho u_1 \\ \rho u_2 \\ \rho u_3 \\ \rho u_1 u_1 + P_{11} \\ \rho u_1 u_2 + P_{12} \\ \rho u_1 u_3 + P_{13} \\ \rho u_2 u_2 + P_{22} \\ \rho u_2 u_3 + P_{23} \\ \rho u_3 u_3 + P_{33} \end{bmatrix} \tag{25}$$

For this transformation we have

$$\varphi'(\mathbf{v}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ u_1 & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ u_2 & 0 & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ u_3 & 0 & 0 & \rho & 0 & 0 & 0 & 0 & 0 & 0 \\ u_1 u_1 & 2\rho u_1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ u_1 u_2 & \rho u_2 & \rho u_1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ u_1 u_3 & \rho u_3 & 0 & \rho u_1 & 0 & 0 & 1 & 0 & 0 & 0 \\ u_2 u_2 & 0 & 2\rho u_2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ u_2 u_3 & 0 & \rho u_3 & \rho u_2 & 0 & 0 & 0 & 0 & 1 & 0 \\ u_3 u_3 & 0 & 0 & 2\rho u_3 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{26}$$

The inverse of the transformation Jacobian is

$$(\varphi')^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -u_1/\rho & 1/\rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -u_2/\rho & 0 & 1/\rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -u_3/\rho & 0 & 0 & 1/\rho & 0 & 0 & 0 & 0 & 0 & 0 \\ u_1 u_1 & -2u_1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ u_1 u_2 & -u_2 & -u_1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ u_1 u_3 & -u_3 & 0 & -u_1 & 0 & 0 & 1 & 0 & 0 & 0 \\ u_2 u_2 & 0 & -2u_2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ u_2 u_3 & 0 & -u_3 & -u_2 & 0 & 0 & 0 & 0 & 1 & 0 \\ u_3 u_3 & 0 & 0 & -2u_3 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{27}$$

References

[1] John M. Greene. Improved Bhatnagar-Gross-Krook model of electron-ion collisions. *The Physics of Fluids*, 16(11):2022–2023, 1973.